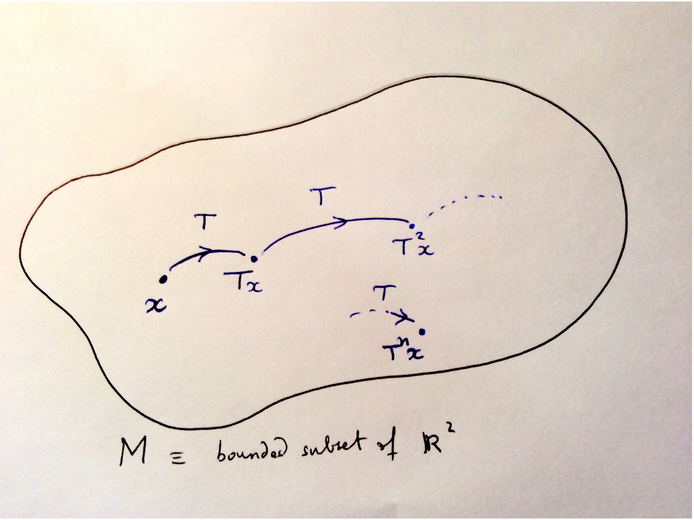
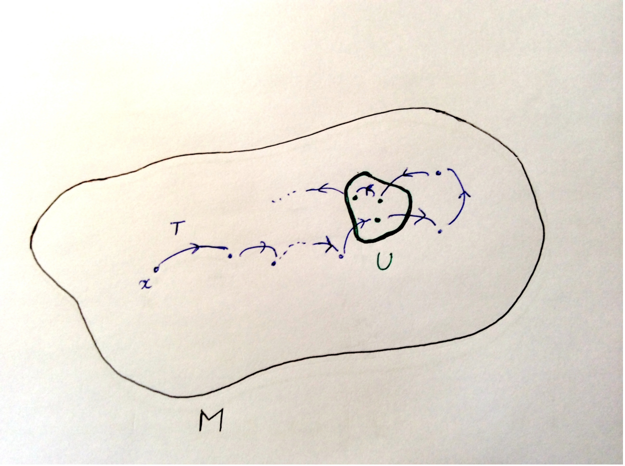
On the Poisson approximation for visits to balls



n iterations of $x \in M$ under the map $T : M \circlearrowleft$



The iterates of x visit $U \subset M$

After a long time, how many times did the orbit of x visit U ?

Missing ingredient: we assume that there is a probability measure μ such that:

$$\mu(T^{-1}A) = \mu(A) \quad \forall\, A \text{ Borel subset of } M.$$

Counting the visits of $x, Tx, \ldots, T^N x$ to U:

$$\sum_{j=0}^{N} \mathbb{1}_U(T^j x).$$

Question:

$$\mu\Big\{x : \sum_{j=0}^{N} \mathbb{1}_U(T^j x) = k\Big\} = ?$$

Natural scale:

$$N \propto \frac{1}{\mu(U)}.$$

Shrinking targets

Sets U such that $\mu(U) \to 0$ in some sense:

- $U = U_n(y)$: cylinder set of a refined partition about y ;
- $U = B_r(y)$: ball centered at y with radius r
- $\dots$

What happens in general ?

- As Roland will explain us:

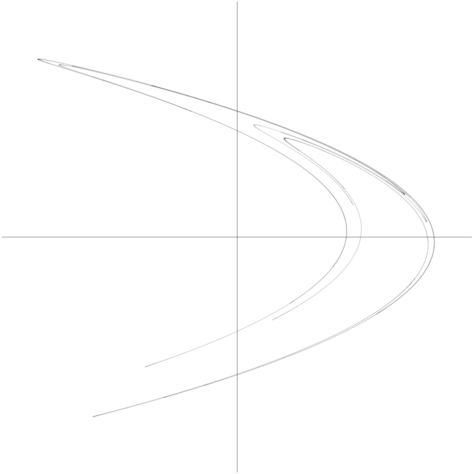
  "you can get anything you want by contructing a suitable $(U_\ell)_\ell$ such that $\mu(U_\ell) \xrightarrow{\ell\to\infty} 0$"

- For **balls** and **cylinders** and for "**chaotic**" dynamical systems, the Poisson law is the natural candidate.

  The rate of mixing should only show up in the "speed" of convergence towards the Poisson law.

Our motivating example: the Hénon map

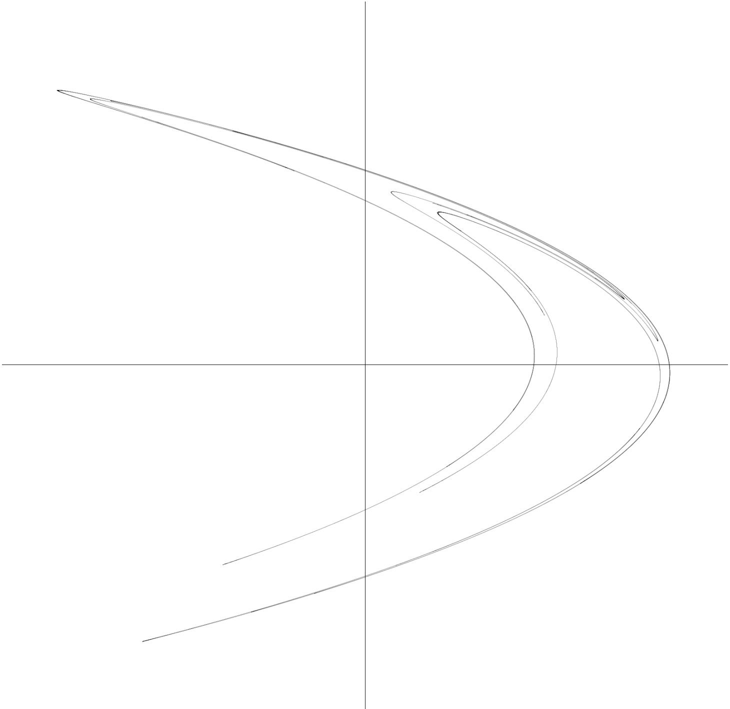
$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y + 1 - ax^2 \\ bx \end{pmatrix}$$



Hénon's attractor

Target set:

$$U = B_r(y)$$



Assumptions

Nonuniformly hyperbolic systems modeled by Young towers such that:

1. their return-time function has an exponential tail;
2. the local unstable manifolds are one-dimensional.

There exists a natural T-invariant probability measure μ (SRB measure) ($\Leftarrow$ the return-time function is integrable).

Theorem (J.-R. C. – P. Collet, 2012)

There exist constants $C, a, b > 0$ such that for all $r \in (0,1)$:

- There exists a set $\mathcal{M}_r$ such that

$$\mu(\mathcal{M}_r) \leq C r^b\,;$$

- For all $y \notin \mathcal{M}_r$ one has

$$\left| \mu \left\{ x \in M \;\middle|\; \sum_{j=0}^{\lfloor t/\mu(B_r(y)) \rfloor} \mathbb{1}_{B_r(y)}(T^j x) = k \right\} - \frac{t^k}{k!}\, e^{-t} \right| \leq C\, r^a$$

for every integer $k \geq 0$ and for every $t > 0$.

Remarks

- this is an approximation result;
- it implies an asymptotic Poisson law as $r \to$ for μ-a.e. center y;
- in fact we control the total variation distance between

$$\sum_{j=0}^{\lfloor t/\mu(B_r(y)) \rfloor} \mathbb{1}_{B_r(y)} \circ T^j \quad \text{and} \quad \text{Poisson}(t)$$

  for $y \in \mathcal{M}_r$. It is $\leq C\ r^a$.
- A central diffulty is to deal with measures which are not absolutely continuous.

Let $(X_n)_{n\in\mathbb{N}}$ be a stationary $\{0,1\}$-valued process and $\varepsilon := \mathbb{P}(X_1 = 1)$.
Then for all positive integers p, M, N such that $M \leq N-1$ and $2 \leq p < N$, one has

$$d_{TV}\big(X_1 + \cdots + X_N, \mathrm{Poisson}(N\varepsilon)\big) \leq R(\varepsilon, N, p, M)$$

The error term $R(\varepsilon, N, p, M)$ is made of three contributions.

One of them is a decorrelation term.

One of them concerns “short returns”.

The last one is independent of the process $(X_n)_{n\in\mathbb{N}}$.

Idea of proof

Take a finite time interval and compare the number of times $X_j = 1$ with the number of times $\tilde{X}_j = 1$ where $(\tilde{X}_n)_{n\in\mathbb{N}}$ is a Bernoulli process such that $\mathbb{P}(\tilde{X}_1 = 1) = \varepsilon$.

Generalizations

1. local unstable manifolds with dimension ≥ 2;
2. polynomial tail for the return-time function.

Results in these directions

Françoise and Benoît (2014) under the assumption

$$\mu\big(B_{r+r^\delta}(x)\backslash B_r(x)\big) = o(\mu(B_r(x)). \tag{1}$$

for $\delta > 1$ not too large.

Examples covered:
solenoid with intermittency and billiard in stadium.

(with a control of the approximation ?)

Haydn and Wasilewska (2014) obtained a Poisson approximation with an error less than $|\log r|^{-\kappa}$ for $\kappa > 0$ small enough. The tail has to decrease with a degree > 4. They also assume something like (1).

Jorge Freitas, Haydn and Nicol (2013) obtained some results for a class of planar billiard maps with polynomial mixing rate (Bunimovich's stadium, flower-like stadia). They have a bound for the error term.

They use inducing.