Asymptotic poissonity for the number of visits to a small ball

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ANR PERTURBATIONS

Work in collaboration with Benoît Saussol

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Poissonity for the visits to a shrinking ball

Let $(\mathcal{M}, d)$ be a Riemanian manifold, $\mathcal{B}$ its Borel σ-algebra, $f : \mathcal{M} \to \mathcal{M}$ preserving the probability measure μ.

- For $x, y \in \mathcal{M}$, $r > 0$ and $m \in \mathbb{Z}_+$, we define

$$\mathcal{N}_{B(x,r)}(m)(y) := \#\{k = 1, ..., m : f^k(y) \in B(x, r)\},$$

with $B(x, r) := \{y \in \mathcal{M} \;:\; d(x, y) < r\}$.

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- <u>Poissonity for $x \in \mathcal{M}$</u> : Let $t > 0$.
  $\mathcal{N}_{B(x,r)}\left(\left\lfloor \frac{t}{\mu(B(x,r))} \right\rfloor\right) \overset{distr.}{\to}_{r\to 0} Y$: Poisson r.v. of mean t.

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- Consequences : $\forall k$, the k-th visit time $T^{(k)}_{B(x,r)}$ to $B(x, r)$ satisfies:
  $\mu(B(x, r))\, T^{(k)}_{B(x,r)} \overset{distr.}{\to}_{r\to 0} X_1 + \cdots + X_k$, where X_j are iid exponential random variables with mean 1. proof

Context : a quick description

Here f is assumed to be invertible (up to a set of μ-measure 0), modeled by a Young tower over a "parallelogram" $\Lambda \subset \mathcal{M}$ with a polynomial tailed return time ([Alves,Azevedo2013])

$$\boxed{\mu(R > n) = \mu(\{y \in \Lambda \, : \, R(y) > n\}) = O(n^{-\zeta}), \ \zeta > 1}.$$

- **Hyperbolicity :**
  a manifold $\gamma^s \subset \mathcal{M}$ is **stable** if $\lim_{n\to+\infty} \mathrm{diam}(f^n\gamma^s) = 0$.
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- We use an argument of [Chazottes,Collet2013] in which a Poissonity result is established for $\mu(R > n) \le C\theta^n$ with $C > 0$ and $\theta \in (0, 1)$.
  Their result applies to the Sinai billiard for which $\mathrm{diam}(f^n\gamma^s) \le C\theta^n$ and $\mathrm{diam}(f^{-n}\gamma^u) \le C\theta^n$.

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- Our result applies to the billiard in the stadium for which
  - the Young tower satisfies $\mu(R > n) = O(n^{-2})$,
  - $\mathrm{diam}(f^n\gamma^s) \le C/n$ and $\mathrm{diam}(f^{-n}\gamma^u) \le C/n$ with $C > 0$.

Examples

- Billiards

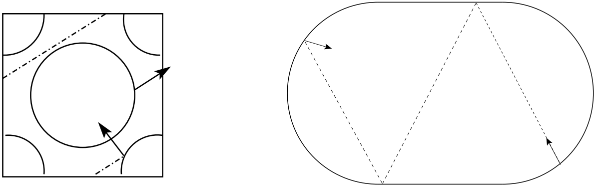

Figure: The Sinai billiard and the stadium billiard

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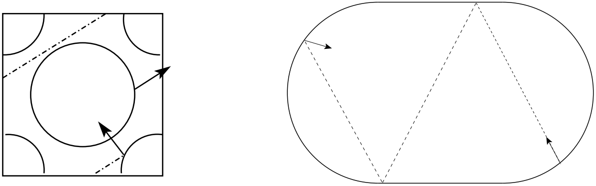

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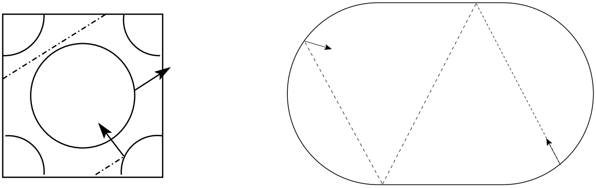

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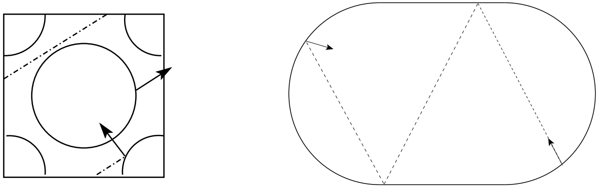

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- other example we treat: the solenoid with intermittency.
- other reference: [Haydn,Wasilewska2014] for $\mu(R > n) = O(n^{-\zeta})$ with ζ large enough.

General Assumptions

- **A good "parallelogram"** $\Lambda = \left(\bigcup_{\gamma^u\in\Gamma^u}\gamma^u\right)\cap\left(\bigcup_{\gamma^s\in\Gamma^s}\gamma^s\right)$ with $\mathrm{Leb}^u_\gamma(\Lambda)>0$ picture
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Theorem([P,Saussol2014]) *Under the previous assumptions, we have the Poissonity for the visits in shrinking balls around μ-a.e. $x \in \mathcal{M}$ satisfying:*

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Proof of the Proposition: adaptation of the proof of the Theorem combined with the fact that for every $x \in \mathcal{M}$ and every $\eta \in (0, 1)$, the assumption on coronas holds along a sequence $(r_n(x))_n$ and more precisely there exists $C(x) > 0$ such that

$$\forall n,\ \ \forall s \in (0, r_n(x)),\ \ \frac{\mu(B(x, r_n(x) + s) \setminus B(x, r_n(x)))}{\mu(B(x, r_n(x)))} \leq \frac{C(x) s^\eta}{(r_n(x))^\eta}.$$

Proof of the general theorem

Due to [Chazottes,Collet2013], the total variation between the Poisson distribution with mean t and the distribution of $\mathcal{N}_{B(x,r)}(N)$ (with $N = \lfloor t/\varepsilon \rfloor$, $\varepsilon := \mu(B(x,r))$) is less than

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- $R_1(\varepsilon, N, p)$ corresponds to the covariance of two indicator functions. Approximating the sets by union of "cylinders" (for the tower) and applying a decorrelation result for these unions, we have:

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- $R_3(\varepsilon, N, p, M) \to 0$ if $M \to +\infty$ and $Mp\varepsilon \to 0$. Conclusion

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- A careful reading of Chernov-Markarian leads to: $\mathrm{diam}(f^n \gamma^s) \leq Cn^{-\alpha}$, $\mathrm{diam}(f^{-n}\gamma^u) \leq Cn^{-\alpha}$, $\alpha = 1$ so $\alpha \cdot d > 1$.

Solenoid with intermittency

- Let $\gamma \in (0,1)$. Let $g : \mathbb{T} \to \mathbb{T}$ be a continuous map of degree ≥ 2 which is C^2 on $\mathbb{T} \setminus \{0\}$, such that $g' > 1$ on $\mathbb{T} \setminus \{0\}$, with $g'(0-) > 1$ and $g(x) = x(1 + ax^\gamma + o(x^\gamma))$ at $0+$.
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- Let $0 < \theta < 1/(1 + \|g'\|_\infty)$. Let $\mathbb{D} := \{z \in \mathbb{C} \ : \ |z| \leq 1\}$. We endow $\mathcal{M} := \mathbb{T} \times \mathbb{D}$ with the max norm. Let $f : \mathcal{M} \to \mathcal{M}$ by $f(x,z) = (g(x), \frac{e^{2i\pi x}}{2} + \theta z)$.

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- We define $\mathcal{M}_0 := \cap_{n \geq 0} f^n \mathcal{M}$. picture
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  diam$(f^n \gamma^s)$, diam$(f^{-n} \gamma^u) \leq Cn^{-\alpha}$, $\alpha = 1 + \frac{1}{\gamma}$ so $\alpha \cdot d > 1$.
- last point: negligeability of coronas.

We want to prove that

$$\exists \delta \in (1, \alpha d),\ \mu(B(X, r + r^\delta) \setminus B(X, r)) = o(\mu(B(X, r)))$$

for μ-a.e. $X \in \mathcal{M}$.
Let $p < \min(2, 1/\gamma)$.
Take a point $X \in \mathcal{M}_0$ such that if $f^{-m}(X) \in [0, 1/2] \times \mathbb{D}$, the series of consecutive integers k containing m such that $f^{-k}(X) \in [0, 1/2] \times \mathbb{D}$ has length in $o(m^{\frac{1}{p}})$
(this is true μ-a.s. due to [Dedecker,Gouëzel,Merlevède2010])

Estimate on coronas for the solenoid

Goal: $\exists \delta \in (1, \alpha d),\ \mu(B(X, r + r^\delta) \setminus B(X, r)) = o(\mu(B(X, r)))$
We consider the intersection of $\mathcal{M}_0$ with the corona $B(x, r + r^\delta) \setminus B(x, r)$.

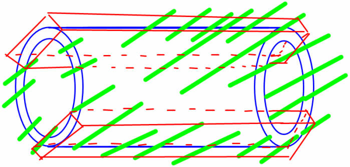
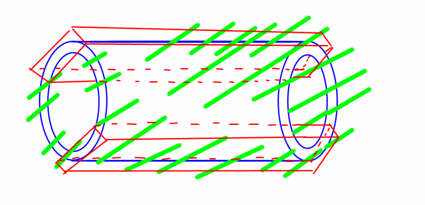


Figure: $\mathcal{M}_0$ in green; non-transversal intersection in red boxes.

the intersection is transversal if angle(z-component of the green spaghetti direction, vertical circle)$> \beta = r^{\delta-1} n^\nu$, with $r \sim \theta^n$, $r^\delta \sim \theta^{n+k}$. We take $p = \min(2, 1/\gamma) - \eta$ with small $\eta > 0$.

Estimate on coronas for the solenoid

Goal: $\exists \delta \in (1, \alpha d)$, $\mu(B(X, r + r^{\delta}) \setminus B(X, r)) = o(\mu(B(X, r)))$
We consider the intersection of $\mathcal{M}_0$ with the corona $B(x, r + r^{\delta}) \setminus B(x, r)$.



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It is true if $\nu < \frac{1}{\gamma}$, $\frac{1}{p} < \nu < \frac{1}{\gamma} + 1 - \frac{1}{\gamma p} - \frac{1}{p}$ so if $\gamma < 1/\sqrt{2}$.