Statistical properties for systems with weak invariant manifolds

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Workshop rare & extreme

Gibbs-Markov-Young structure

Let M be a finite dimensional Riemannian compact manifold and consider a diffeomorphism $f : M \to M$. Let $\Lambda \subset M$ have a hyperbolic product structure.

Definition

$\Lambda_1 \subseteq \Lambda$ is an **s-subset** if Λ_1 satisfies:

- it has a hyperbolic product structure;
- its defining families Γ_1^s and Γ_1^u can be chosen with $\Gamma_1^s \subseteq \Gamma^s$ and $\Gamma_1^u = \Gamma^u$.

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Definition

Λ has a **Gibbs-Markov-Young (GMY) structure** if it has a hyperbolic product structure and the following properties (P_0)-(P_5) hold.

(P_0) **Lebesgue detectable**
There exists an unstable manifold $\gamma \in \Gamma^u$ such that $\mathrm{Leb}_\gamma(\Lambda \cap \gamma) > 0$.

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(P_1) **Markov property**
There are pairwise disjoint s-subsets $\Lambda_1, \Lambda_2, \ldots \subseteq \Lambda$ such that:

(a) $\mathrm{Leb}_\gamma\left((\Lambda \setminus \bigcup_{i=1}^{\infty} \Lambda_i) \bigcap \gamma^u\right) = 0$ on each $\gamma^u \in \Gamma^u$;

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(b) for each $i \in \mathbb{N}$ there exists a $R_i \in \mathbb{N}$ such that $f^{R_i}(\Lambda_i)$ is an u-subset and, for all $x \in \Lambda_i$,

$$f^{R_i}(\gamma^s(x)) \subseteq \gamma^s(f^{R_i}(x)) \quad \text{and} \quad f^{R_i}(\gamma^u(x)) \supseteq \gamma^u(f^{R_i}(x)).$$

For the remaining properties we assume that $C > 0$, $\alpha > 1$ and $0 < \beta < 1$ are constants depending only on f and Λ.

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(P$_2$) **Polynomial contraction on stable leaves**

$$\forall\, y \in \gamma^s(x)\ \forall\, n \in \mathbb{N} \quad d(f^n(x), f^n(y)) \leq \frac{C}{n^\alpha} d(x,y).$$

(P$_3$) **Backward polynomial contraction on unstable leaves**

$$\forall\, y \in \gamma^u(x)\ \forall\, n \in \mathbb{N} \quad d(f^{-n}(x), f^{-n}(y)) \leq \frac{C}{n^\alpha} d(x,y).$$

Define a **return time function** $R : \Lambda \to \mathbb{N}$ and a **return function** $f^R : \Lambda \to \Lambda$ as

$$R|_{\Lambda_i} = R_i \quad \text{and} \quad f^R|_{\Lambda_i} = f^{R_i}|_{\Lambda_i}.$$

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For $x, y \in \Lambda$, let the **separation time** $s(x, y)$ be defined as

$$s(x,y) = \min\left\{n \in \mathbb{N}_0 : (f^R)^n(x) \text{ and } (f^R)^n(y) \text{ are in distinct } \Lambda_i\right\}.$$

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(P$_4$) **Bounded distortion**
For $\gamma \in \Gamma^u$ and $x, y \in \Lambda \cap \gamma$

$$\log \frac{\det D(f^R)^u(x)}{\det D(f^R)^u(y)} \leq C \beta^{s(f^R(x), f^R(y))}.$$

(P_5) **Regularity of the stable foliation**

For each $\gamma, \gamma' \in \Gamma^u$, define

$$\begin{array}{rccc} \Theta_{\gamma',\gamma}: & \gamma' \cap \Lambda & \to & \gamma \cap \Lambda \\ & x & \mapsto & \gamma^s(x) \cap \gamma. \end{array}$$

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(a) Θ is absolutely continuous and

$$\frac{d(\Theta_* \operatorname{Leb}_{\gamma'})}{d \operatorname{Leb}_\gamma}(x) = \prod_{n=0}^{\infty} \frac{\det\ Df^u(f^n(x))}{\det\ Df^u(f^n(\Theta^{-1}(x)))}.$$

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(b) Denoting

$$u(x) = \frac{d(\Theta_* \operatorname{Leb}_{\gamma'})}{d \operatorname{Leb}_\gamma}(x),$$

we have

$$\forall\, x, y \in \gamma' \cap \Lambda \quad \log \frac{u(x)}{u(y)} \leq C \beta^{s(x,y)}.$$

Statement of results

Let μ be an SRB-measure.

Definition

Given $n \in \mathbb{N}$, we define the **correlation of observables** $\varphi, \psi \in H_\eta$ as

$$\mathcal{C}_n(\varphi, \psi, \mu) = \left| \int (\varphi \circ f^n)\psi \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu \right|.$$

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Definition

If μ is an ergodic probability measure and $\varepsilon > 0$, the **large deviation** at time n of the time average of the observable ϕ from its spatial average is given by

$$LD(\phi, \varepsilon, n, \mu) = \mu\left\{ \left| \frac{1}{n} \sum_{i=1}^{n-1} \phi \circ f^i - \int \phi \, d\mu \right| > \varepsilon \right\}.$$

Theorem (A)

Suppose that f admits a GMY structure Λ with $\gcd\{R_i\} = 1$ for which there are $\gamma \in \Gamma^u$, $\zeta > 1$ and $C_1 > 0$ such that

$$\mathrm{Leb}_\gamma\{R > n\} \leq \frac{C_1}{n^\zeta}.$$

Then, given $\varphi, \psi \in H_\eta$, there exists $C_2 > 0$ such that for every $n \geq 1$

$$\mathcal{C}_n(\varphi, \psi, \mu) \leq C_2 \max\left\{\frac{1}{n^{\zeta-1}}, \frac{1}{n^{\alpha\eta}}\right\},$$

where $\alpha > 0$ is the constant in (P_2) and (P_3).

Statement of results

Theorem (B)

Suppose that f admits a GMY structure Λ with $\gcd\{R_i\} = 1$ for which there are $\gamma \in \Gamma^u$, $\zeta > 1$ and $C_1 > 0$ such that

$$\mathrm{Leb}_\gamma\{R > n\} \le \frac{C_1}{n^\zeta}.$$

Then there are $\eta_0 > 0$ and $\zeta_0 = \zeta_0(\eta_0) > 1$ such that for all $\eta > \eta_0$, $1 < \zeta < \zeta_0$, $\varepsilon > 0$, $p > \max\{1, \zeta - 1\}$ and $\phi \in \mathcal{H}_\eta$, there exists $C_2 > 0$ such that for every $n \ge 1$

$$LD(\phi, \varepsilon, n, \mu) \le \frac{C_2}{\varepsilon^{2p}} \frac{1}{n^{\zeta-1}}.$$

Tower structure

Define a **tower**

$$\Delta = \{(x,l) : x \in \Lambda \text{ and } 0 \leq l < R(x)\}$$

and a **tower map** $F : \Delta \to \Delta$ as

$$F(x,l) = \begin{cases} (x, l+1) & \text{if } l+1 < R(x), \\ (f^R(x), 0) & \text{if } l+1 = R(x). \end{cases}$$

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The set the ***l*-th level of the tower** is defined as

$$\Delta_l = \{(x, l) \in \Delta\}.$$

Note that Δ_0 is naturally identified with Λ.

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Note that Δ_0 is naturally identified with Λ.
Define a projection map

$$\begin{array}{rccc} \pi : & \Delta & \to & \bigcup_{n=0}^{\infty} f^n(\Delta_0) \\ & (x,l) & \mapsto & f^l(x) \end{array}$$

and observe that $f \circ \pi = \pi \circ F$.

Tower structure

- Let $\Delta_{0,i} = \Lambda_i$ and

$$\Delta_{l,i} = \{(x,l) \in \Delta_l : x \in \Delta_{0,i}\}.$$

$\mathcal{Q} = \{\Delta_{l,i}\}_{l,i}$ is a partition of Δ.

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- Introduce a sequence of partitions $(\mathcal{Q}_n)$ of Δ defined as

$$\mathcal{Q}_0 = \mathcal{Q} \quad \text{and} \quad \mathcal{Q}_n = \bigvee_{i=0}^{n} F^{-i}\mathcal{Q} \text{ for } n \in \mathbb{N}.$$

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- Define $\sim$ on Λ by $x \sim y$ if $y \in \gamma^s(x)$ and consider the **quotient tower** $\bar{\Delta}$, set $\bar{\Delta}_l = \Delta_l/\sim$ and $\bar{\Delta}_{l,i} = \Delta_{l,i}/\sim$.

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- Define $\bar{F} : \bar{\Delta} \to \bar{\Delta}$ and partitions $\bar{\mathcal{Q}}$ and $(\bar{\mathcal{Q}}_n)$ of $\bar{\Delta}$ analogously to the previous definition.

Tower structure

- The definitions of the **return time** $\bar{R} : \bar{\Delta}_0 \to \mathbb{N}$ and the **separation time** $\bar{s} : \bar{\Delta}_0 \times \bar{\Delta}_0 \to \mathbb{N}$ are induced by the definitions in Δ_0.

Tower structure

- The definitions of the **return time** $\bar{R} : \bar{\Delta}_0 \to \mathbb{N}$ and the **separation time** $\bar{s} : \bar{\Delta}_0 \times \bar{\Delta}_0 \to \mathbb{N}$ are induced by the definitions in Δ_0.
- Extend the **separation time** $\bar{s}$ to $\bar{\Delta} \times \bar{\Delta}$ as follows:
  - if $x, y \in \bar{\Delta}_{l,i}$, take $\bar{s}(x, y) = \bar{s}(x_0, y_0)$, where x_0, y_0 are the corresponding elements of $\bar{\Delta}_{0,i}$;
  - otherwise, take $\bar{s}(x, y) = 0$.

Sketch of the proof of Theorem A

Lemma

There exists $C > 0$ such that, for all $k \in \mathbb{N}$ and $Q \in \mathcal{Q}_{2k}$,

$$\operatorname{diam}(\pi F^k(Q)) \leq \frac{C}{k^\alpha}.$$

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- There is a measure ν such that $\mu = \pi_*\nu$ and $\bar{\nu} = \bar{\pi}_*\nu$.
- For $\varphi, \psi \in H_\eta$, let $\widetilde{\varphi} = \varphi \circ \pi$ and $\widetilde{\psi} = \psi \circ \pi$.

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- There is a measure ν such that $\mu = \pi_* \nu$ and $\bar{\nu} = \bar{\pi}_* \nu$.
- For $\varphi, \psi \in H_\eta$, let $\widetilde{\varphi} = \varphi \circ \pi$ and $\widetilde{\psi} = \psi \circ \pi$.
- We can easily verify that $\mathcal{C}_n(\varphi, \psi, \mu) = \mathcal{C}_n(\widetilde{\varphi}, \widetilde{\psi}, \nu)$.

Given $n \in \mathbb{N}$, fix $k < n/2$ and define

$$\bar{\varphi}_k|_Q = \inf\{\widetilde{\varphi} \circ F^k(x) : x \in Q\}, \qquad \text{for } Q \in \mathcal{Q}_{2k}.$$

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$$\bar{\varphi}_k|_Q = \inf\{\widetilde{\varphi} \circ F^k(x) : x \in Q\}, \qquad \text{for } Q \in \mathcal{Q}_{2k}.$$

Lemma

For $\varphi, \psi \in H_\eta$, let $\widetilde{\varphi}$, $\widetilde{\psi}$ and $\bar{\varphi}_k$ be defined as above. Then

$$|\mathcal{C}_n(\widetilde{\varphi}, \widetilde{\psi}, \nu) - \mathcal{C}_{n-k}(\bar{\varphi}_k, \widetilde{\psi}, \nu)| \leq \frac{C_2}{k^{\alpha\eta}},$$

Define $\bar{\psi}_k$ in a similar way to $\bar{\varphi}_k$. Let $\bar{\psi}_k\nu$ and $\widetilde{\psi}_k$ be the signed measures such that

$$\frac{d\bar{\psi}_k\nu}{d\nu} = \bar{\psi}_k \quad \text{and} \quad \widetilde{\psi}_k = \frac{dF_*^k\bar{\psi}_k\nu}{d\nu}.$$

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Lemma

For $\varphi, \psi \in H_\eta$, let $\bar{\varphi}_k$, $\widetilde{\psi}$ and $\widetilde{\psi}_k$ be defined as before. Then

$$\left|\mathcal{C}_{n-k}(\bar{\varphi}_k, \widetilde{\psi}, \nu) - \mathcal{C}_{n-k}(\bar{\varphi}_k, \widetilde{\psi}_k, \nu)\right| \leq \frac{C_3}{k^{\alpha\eta}},$$

Sketch of the proof of Theorem A

Lemma

For $\varphi, \psi \in H_\eta$, let $\bar{\varphi}_k$, $\widetilde{\psi}_k$ and $\bar{\psi}_k$ be defined as before. Then

$$\mathcal{C}_{n-k}(\bar{\varphi}_k, \widetilde{\psi}_k, \nu) = \mathcal{C}_n(\bar{\varphi}_k, \bar{\psi}_k, \bar{\nu}).$$

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$$\mathcal{C}_{n-k}(\bar{\varphi}_k, \widetilde{\psi}_k, \nu) = \mathcal{C}_n(\bar{\varphi}_k, \bar{\psi}_k, \bar{\nu}).$$

Let $\bar{\lambda}_k$ be a certain measure that is defined depending on $\bar{\psi}_k$.

Lemma

For $\varphi, \psi \in H_\eta$, let $\bar{\varphi}_k$ and $\bar{\psi}_k$ be defined as before. Then

$$\mathcal{C}_n(\bar{\varphi}_k, \bar{\psi}_k, \bar{\nu}) \leq C_4 |\bar{F}_*^{n-2k} \bar{\lambda}_k - \bar{\nu}|,$$

Sketch of the proof of Theorem A

Given $0 < \beta < 1$, we define

$$\mathcal{F}_\beta = \left\{\varphi : \bar{\Delta} \to \mathbb{R} : \exists\, C_\varphi > 0 \;\forall\, x, y \in \bar{\Delta} \quad |\varphi(x) - \varphi(y)| \le C_\varphi \beta^{\bar{s}(x,y)}\right\}$$

$$\mathcal{F}_\beta^+ = \Big\{\varphi \in \mathcal{F}_\beta : \exists\, C_\varphi > 0 \text{ such that on each } \bar{\Delta}_{l,i}, \text{ either } \varphi \equiv 0$$

$$\text{or } \varphi > 0 \text{ and for all } x, y \in \bar{\Delta}_{l,i} \quad \left|\frac{\varphi(x)}{\varphi(y)} - 1\right| \le C_\varphi \beta^{\bar{s}(x,y)}\Big\}$$

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$$\left. \text{or } \varphi > 0 \text{ and for all } x, y \in \bar{\Delta}_{l,i} \quad \left|\frac{\varphi(x)}{\varphi(y)} - 1\right| \leq C_\varphi \beta^{\bar{s}(x,y)}\right\}$$

Property (P_5) enables us to define a certain measure $\bar{m}$ in Δ.

Theorem (Young)

For $\varphi \in \mathcal{F}_\beta^+$ let $\bar{\lambda}$ be the measure whose density with respect to $\bar{m}$ is φ. If $\mathrm{Leb}\{\bar{R} > n\} \leq Cn^{-\zeta}$, for some $C > 0$ and $\zeta > 1$, then there is $C' > 0$ such that

$$\left|\bar{F}_*^n \bar{\lambda} - \bar{\nu}\right| \leq C' n^{-\zeta+1}.$$

Moreover, C' depends only on C_φ.

Sketch of the proof of Theorem B

Given $\theta > 0$, we define

$$\mathcal{G}_\theta = \left\{ \varphi : \Delta \to \mathbb{R} : \exists\, c_\varphi > 0\, \forall x, y \in \Delta \quad |\varphi(x) - \varphi(y)| \leq \frac{c_\varphi}{\max\{s(x,y),1\}^\theta} \right\}$$

$$\mathcal{G}_\theta^+ = \Big\{ \varphi \in \mathcal{G}_\theta : \exists\, c_\varphi > 0 \text{ such that on each } \Delta_{l,i}, \text{ either } \varphi \equiv 0 \text{ or}$$

$$\varphi > 0 \text{ and for all } x, y \in \Delta_{l,i} \quad \left| \frac{\varphi(x)}{\varphi(y)} - 1 \right| \leq \frac{c_\varphi}{\max\{s(x,y),1\}^\theta} \Big\}$$

Sketch of the proof of Theorem B

Theorem

Assume that there is $C > 0$ *such that*

$$m\{\bar{R} > n\} \leq \frac{C}{n^{\zeta}}.$$

Then there are $\theta_0 > 1$ *and* $1 < \zeta_0 = \zeta_0(\theta)$ *such that for all* $\theta \geq \theta_0$ *and* $1 < \zeta < \zeta_0$*, given* $\varphi \in \mathcal{G}_\theta^+$ *there exists* $C' > 0$*, depending only on* D_φ*, such that*

$$\left|\bar{F}_*^n \bar{\lambda} - \bar{\nu}\right| \leq \frac{C'}{n^{\zeta-1}},$$

where $\bar{\lambda}$ *be the measure whose density with respect to* $\bar{m}$ *is* φ*.*

Corollary

Assume that there is $C > 0$ *such that*

$$m\{\bar{R} > n\} \leq \frac{C}{n^{\zeta}}.$$

Then there are $\theta_0 > 1$ *and* $1 < \zeta_0 = \zeta_0(\theta)$ *such that for all* $\theta \geq \theta_0$ *and* $1 < \zeta < \zeta_0$*, given* $\varphi \in \mathcal{G}_\theta$ *and* $\psi \in L^\infty$ *there exists* $C' > 0$*, depending only on* D_φ *and* $\|\psi\|_\infty$*, such that*

$$\mathcal{C}_n(\psi, \varphi, \bar{\nu}) \leq \frac{C'}{n^{\zeta-1}},$$

and let $\bar{\lambda}$ *be the measure whose density with respect to* $\bar{m}$ *is* φ.

Proposition

Let f has a GMY structure Λ and $\phi : M \to \mathbb{R}$ be a function belonging to $\mathcal{H}_\eta$ for $\eta > 1/\alpha$. Then there exist functions $\chi, \psi : \Delta \to \mathbb{R}$ such that:

1. $\chi \in L^\infty(\Delta)$ *and* $\|\chi\|_\infty$ *depends only on* $|\phi|_\eta$*;*
2. $\phi \circ \pi = \psi + \chi - \chi \circ F$*;*
3. ψ *depends only on future coordinates;*
4. *the function* $\psi : \bar{\Delta} \to \mathbb{R}$ *belongs to* $\mathcal{G}_\theta$*, for* $\theta = \alpha\,\eta - 1$*.*

Sketch of the proof of Theorem B

Proposition

Let $\zeta > 0$ *and* $\psi \in \mathcal{G}_\theta(\bar{\Delta})$*, for some* $\theta > 0$*. Suppose there exists* $C_4 > 0$ *such that, for all* $w \in L^\infty(\bar{\Delta})$ *and all* $n \geq n_0$ *we have*

$$C_n(w, \psi, \bar{\nu}) \leq \frac{C_4}{n^\zeta},$$

where C_4 *depends only on* c_ψ *and* $\|w\|_\infty$*. Then, for* $\varepsilon > 0$ *and* $p > \max\{1, \zeta\}$,

$$LD(\psi, \varepsilon, n, \bar{\nu}) \leq \frac{C_5}{\varepsilon^{2p} n^\zeta}.$$

Tower product

In this section we simplify the notations by removing all bars.

- Let $\Phi = \varphi \times \varphi'$.
- Let λ and λ' be probability measures in Δ whose densities

$$\varphi = \frac{d\lambda}{dm} \quad \text{and} \quad \varphi' = \frac{d\lambda'}{dm}$$

  belong to $\mathcal{G}_\theta^+$.

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- Consider

$$\begin{array}{rccc} F \times F : & \Delta \times \Delta & \to & \Delta \times \Delta \\ & (x, y) & \mapsto & (F(x), F(y)) \end{array}$$

and the measure $P = \lambda \times \lambda'$ in $\Delta \times \Delta$.

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- Consider

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  and the measure $P = \lambda \times \lambda'$ in $\Delta \times \Delta$.
- Let $\pi, \pi' : \Delta \times \Delta \to \Delta$ be the projections on the first and second coordinates.

Tower product

- We define a simultaneous return time $T : \Delta \times \Delta \longrightarrow \mathbb{N}$.
- Define a sequence of *stopping times* in $\Delta \times \Delta$, $0 \equiv T_0 < T_1 < \cdots$, as

$$T_1 = T \text{ and } T_n = T_{n-1} + T \circ (F \times F)^{T_{n-1}}, \text{ for } n \geq 2.$$

Tower product

- We define a simultaneous return time $T : \Delta \times \Delta \longrightarrow \mathbb{N}$.
- Define a sequence of *stopping times* in $\Delta \times \Delta$, $0 \equiv T_0 < T_1 < \cdots$, as

$$T_1 = T \text{ and } T_n = T_{n-1} + T \circ (F \times F)^{T_{n-1}}, \text{ for } n \geq 2.$$

- Let $\widehat{F} = (F \times F)^T : \Delta \times \Delta \longrightarrow \Delta \times \Delta$.
- It is easy to verify that

$$\forall n \in \mathbb{N} \quad \widehat{F}^n = (F \times F)^{T_n}.$$

Tower product

- Let $(\widehat{\xi}_i)$ be a certain sequence of partition on $\Delta \times \Delta$ and $\widehat{\xi}_i(z)$ is the element of $\widehat{\xi}_i$ which contains z.
- We will find a sequence of densities $(\widehat{\Phi}_i)$ in $\Delta \times \Delta$ such that
  - $\widehat{\Phi}_0 \geq \widehat{\Phi}_1 \geq \cdots ;$
  - for all $i \in \mathbb{N}$ and $\widehat{\Gamma} \in \widehat{\xi}_i$,

$$\pi_* \widehat{F}^i_* \big((\widehat{\Phi}_{i-1} - \widehat{\Phi}_i)((m \times m)|\widehat{\Gamma}) \big) = \pi'_* \widehat{F}^i_* \big((\widehat{\Phi}_{i-1} - \widehat{\Phi}_i)((m \times m)|\widehat{\Gamma}) \big).$$

- Define, for $i < i_0$, $\widehat{\Phi}_i \equiv \Phi$ and, for $i \geq i_0$,

$$\widehat{\Phi}_i(z) = \left(\frac{\widehat{\Phi}_{i-1}(z)}{J\widehat{F}^i(z)} - \varepsilon_i \min_{w \in \widehat{\xi}_i(z)} \frac{\widehat{\Phi}_{i-1}(w)}{J\widehat{F}^i(w)} \right) J\widehat{F}^i(z),$$

where

$$\varepsilon_i = e^K \big(1 - (\frac{i-1}{i})^\rho\big).$$

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Lemma

Assume that $\theta > e^K \rho$. Then, there exists $i_0 \in \mathbb{N}$ such that, for $i \geq i_0$, we have

$$\widehat{\Phi}_i \leq \left(\frac{i-1}{i}\right)^\rho \widehat{\Phi}_{i-1} \quad in \quad \Delta \times \Delta.$$

An example

Let:

- f_0 be an orientation preserving C^2 Anosov diffeomorphism of the torus;
- $W_0, \ldots, W_d$ be a Markov partition for f_0, fixed point $(0,0)$ in the interior of W_0;
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Let:

- f_0 be an orientation preserving C^2 Anosov diffeomorphism of the torus;
- $W_0, \ldots, W_d$ be a Markov partition for f_0, fixed point $(0,0)$ in the interior of W_0;
- the transition matrix A of f_0 be **aperiodic**;
- f_0 have a product form in W_0, i.e., $f_0(a,b) = \big(\phi_0(a), \psi_0(b)\big)$;
- ϕ_0 and ψ_0 be orientation preserving;
- there exist $\lambda > 1$ such that $\phi_0' > \lambda$ and $0 < \psi_0' < 1/\lambda$;
- the local stable manifold of $(0,0)$ be $\{a = 0\}$ and the local unstable manifold of $(0,0)$ be $\{b = 0\}$.

An example

- We want f to be a perturbation of f_0 such that:
  - f also has the product structure in W_0, i.e., $f(a,b) = \big(\phi(a), \psi(b)\big)$.
  - the local stable and unstable manifolds of $(0,0)$ are also $\{a=0\}$ and $\{b=0\}$, respectively.
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  - $(0,0)$ is a fixed point of f with $\phi'(0) = 1 = \psi'(0)$.
- In a rectangular neighbourhood of $(0,0)$ contained in W_0, choosing $0 < \theta < 1$,

$$\phi(a) = a(1 + a^\theta), \qquad \psi(b) = \phi^{-1}(b)$$

  and ϕ and ψ coincide with ϕ_0 and ψ_0, respectively, near the boundary of W_0.
- In $\mathbb{T}^2 \setminus W_0$, $f(a,b) = f_0(a,b)$.

An example

Choosing $\Lambda = W_1$, we have

- f satisfies the properties (P_0)-(P_5), in particular,

Proposition

There exists $C > 0$ such that for all $n \in \mathbb{N}$ and $x, y \in \gamma^u \in \Gamma^u$ we have

(a) $d(f^{-n}(x), f^{-n}(y)) \leq \frac{C}{n^{1+1/\theta}} d(x, y)$;

(b) $d(f^{n}(x), f^{n}(y)) \leq \frac{C}{n^{1+1/\theta}} d(x, y)$.

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- there is a polynomial decay of the recurrence times to some unstable leaf on W_1, i.e.,

Proposition

There exists $C > 0$ such that, for sufficiently large n,

$$\mathrm{Leb}_\gamma\{R > n\} \leq \frac{C}{n^{1+1/\theta}}.$$

An example

Therefore, f is in the conditions of Theorems (A) and (B):

Theorem

Let f be as above and take $\varphi, \psi \in H_\eta$. Then, there exists $C_2 > 0$ such that for every $n \geq 1$,

- $\mathcal{C}_n(\varphi, \psi, \mu) \leq \dfrac{C_2}{n^{1/\theta}}$ *if* $\eta > \frac{1}{\theta+1}$;
- $\mathcal{C}_n(\varphi, \psi, \mu) \leq \dfrac{C_2}{n^{(1+1/\theta)\eta}}$ *if* $\eta \leq \frac{1}{\theta+1}$.

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Theorem

Let f be as above. There are $\eta_0 > 0$ and $\zeta_0 = \zeta_0(\eta_0) > 1$ such that for all $\eta > \eta_0$, $1 < \zeta < \zeta_0$, $\varepsilon > 0$, $p > 1/\theta$ and $\phi \in \mathcal{H}_\eta$, there exists $C_2 > 0$ such that for every $n \geq 1$

$$LD(\phi, \varepsilon, n, \mu) \leq \frac{C_2}{\varepsilon^{2p}} \frac{1}{n^{1/\theta}}.$$

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