# Periodicity and clustering of extreme events

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## Extreme Value Theory

Consider a stationary stochastic process $X_0, X_1, X_2, \ldots$ with marginal d.f. $F$.

Let $\bar{F} = 1 - F$ and $u_F = \sup\{x : F(x) < 1\}$.

The main goal of the Extreme Value Theory (EVT) is the study of the distributional properties of the maximum

$$M_n = \max\{X_0, \ldots, X_{n-1}\} \tag{1}$$

as $n \to \infty$.

**Definition**

We say that we have an Extreme value law (EVL) for $M_n$ if there is a non-degenerate d.f. $H : \mathbb{R} \to [0,1]$ (with $H(0) = 0$) and for all $\tau > 0$, there exists a sequence of levels $u_n = u_n(\tau)$ such that

$$nP(X_0 > u_n) \to \tau \text{ as } n \to \infty, \tag{2}$$

and for which the following holds:

$$P(M_n \le u_n) \to \bar{H}(\tau) \text{ as } n \to \infty. \tag{3}$$

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## The independent case

In the case $X_0, X_1, X_2, \ldots$ are i.i.d. r.v. then since

$$P(M_n \leq u_n) = (F(u_n))^n$$

we have that if (2) holds, then (3) holds with $\bar{H}(\tau) = e^{-\tau}$:

$$P(M_n \leq u_n) = (1 - P(X_0 > u_n))^n \sim \left(1 - \frac{\tau}{n}\right)^n \to e^{-\tau} \text{ as } n \to \infty,$$

and vice-versa.

When $X_0, X_1, X_2, \ldots$ are not i.i.d. but satisfy some mixing condition $D(u_n)$ introduced by Leadbetter then something can still be said about $H$.

# Condition $D(u_n)$ from Leadbetter

Let $F_{i_1,\dots,i_n}$ denote the joint d.f. of $X_{i_1},\dots,X_{i_n}$, and set $F_{i_1,\dots,i_n}(u) = F_{i_1,\dots,i_n}(u,\dots,u)$.

**Condition ($D(u_n)$)**

We say that $D(u_n)$ holds for the sequence $X_0, X_1, \dots$ if for any integers $i_1 < \dots < i_p$ and $j_1 < \dots < j_k$ for which $j_1 - i_p > t$, and any large $n \in \mathbb{N}$,

$$\left|F_{i_1,\dots,i_p,j_1,\dots,j_k}(u_n) - F_{i_1,\dots,i_p}(u_n)F_{j_1,\dots,j_k}(u_n)\right| \le \gamma(n,t),$$

where $\gamma(n,t_n) \xrightarrow[n\to\infty]{} 0$, for some sequence $t_n = o(n)$.

**Theorem ([C81], see also [LLR83])**

*If $D(u_n)$ holds for $X_0, X_1, \ldots$ and the limit* (3) *exists for some $\tau > 0$ then there exists $0 \leq \theta \leq 1$ such that $\bar{H}(\tau) = e^{-\theta\tau}$ for all $\tau > 0$.*

**Definition**

We say that $X_0, X_1, \ldots$ has an *Extremal Index* (EI) $0 \leq \theta \leq 1$ if we have an EVL for $M_n$ with $\bar{H}(\tau) = e^{-\theta\tau}$ for all $\tau > 0$.

# Linear normalising sequences

The sequences of real numbers $u_n = u_n(\tau)$, $n = 1, 2, \ldots$, are usually taken to be one parameter linear families such as $u_n = a_n y + b_n$, where $y \in \mathbb{R}$ and $a_n > 0$, for all $n \in \mathbb{N}$.

Observe that $\tau$ depends on $y$ through $u_n$ and, in fact, depending on the tail of the marginal d.f. $F$, we have that $\tau = \tau(y)$ is of one of the following 3 types (for some $\alpha > 0$):

Type 1: $\tau_1(y) = e^{-y}$ for $y \in \mathbb{R}$,

Type 2: $\tau_2(y) = y^{-\alpha}$ for $y > 0$,

Type 3: $\tau_3(y) = (-y)^{\alpha}$ for $y \leq 0$.

## Characterization of the three types

**Theorem (Gnedenko)**

*Necessary and sufficient conditions for $\tau$ to be of one of the three types are:*

*Type 1: There exists some strictly positive function g such that, for all real y,*

$$\lim_{t \uparrow u_F} \frac{1 - F(t + yg(t))}{1 - F(t)} = e^{-y}.$$

*Type 2:* $u_F = \infty$ *and* $\lim_{t \to \infty}(1 - F(ty))/(1 - F(t)) = y^{-\alpha}, \alpha > 0$, *for each* $y > 0$.

*Type 3:* $u_F < \infty$ *and* $\lim_{h \downarrow 0}(1 - F(u_F - yh))/(1 - F(u_F - h)) = y^{\alpha}, \alpha > 0$, *for each* $y > 0$.

**Corollary**

*The constants $a_n$ and $b_n$ may be taken as follows:*

*Type 1:* $a_n = g(\gamma_n)$, $b_n = \gamma_n$;

*Type 2:* $a_n = \gamma_n$, $b_n = 0$;

*Type 3:* $a_n = u_F - \gamma_n$, $b_n = u_F$,

*where* $\gamma_n = F^{-1}(1 - 1/n) = \inf\{x : F(x) \geq 1 - 1/n\}$.

## Examples

1. If $F(x) = 1 - e^{-x}$ then $\tau$ is of type 1.

2. If $F(x) = 1 - kx^{-\alpha}$, $\alpha > 0$, $K > 0$, $x \geq K^{1/\alpha}$, then $\tau$ is of type 2.

3. If $F(x) = x$, $0 \leq x \leq 1$, then $\tau$ is of type 3.

## Hitting Times and Kac's Lemma

Consider the system $(\mathcal{X}, \mathcal{B}, \mu, f)$, where $\mathcal{X}$ is a topological space, $\mathcal{B}$ is the Borel $\sigma$-algebra, $f : \mathcal{X} \to \mathcal{X}$ is a measurable map and $\mu$ is an $f$-invariant probability measure, *i.e.*, $\mu(f^{-1}(B)) = \mu(B)$, for all $B \in \mathcal{B}$.

For a set $A \subset \mathcal{X}$ let $r_A(x)$ the *first hitting time to A* of the point $x$, i.e. $r_A(x) = \min\{j \in \mathbb{N} : f^j(x) \in A\}$.

Let $\mu_A$ denote the conditional measure on $A$, i.e. $\mu_A := \frac{\mu|_A}{\mu(A)}$.

By Kac's Lemma, the expected value of $r_A$ with respect to $\mu_A$ is

$$\int_A r_A \, d\mu_A = 1/\mu(A).$$

### Definition

Given a sequence of sets $(U_n)_{n\in\mathbb{N}}$ so that $\mu(U_n)\to 0$, the system has *RTS* $\tilde{G}$ for $(U_n)_{n\in\mathbb{N}}$ if for all $t\geq 0$

$$\mu_{U_n}\left(r_{U_n}\leq\frac{t}{\mu(U_n)}\right)\to\tilde{G}(t)\ \text{ as } n\to\infty. \tag{4}$$

and the system has *HTS* $G$ for $(U_n)_{n\in\mathbb{N}}$ if for all $t\geq 0$

$$\mu\left(r_{U_n}\leq\frac{t}{\mu(U_n)}\right)\to G(t)\ \text{ as } n\to\infty, \tag{5}$$

## Stationary stochastic processes arising from chaotic dynamics

Consider a discrete dynamical system

$$(\mathcal{X}, \mathcal{B}, \mu, f),$$

where
$\mathcal{X}$ is a $d$-dimensional Riemannian manifold,
$\mathcal{B}$ is the Borel $\sigma$-algebra,
$f : \mathcal{X} \to \mathcal{X}$ is a map,
$\mu$ is an $f$-invariant probability measure.

In this context, we consider the stochastic process $X_0, X_1, \ldots$ given by

$$X_n = \varphi \circ f^n, \quad \text{for each } n \in \mathbb{N}, \tag{6}$$

where $\varphi : \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}$ is an observable (achieving a global maximum at $\xi \in \mathcal{X}$) of the form

$$\varphi(x) = g\left(\mu(B_{\text{dist}(x,\zeta)}(\zeta))\right), \tag{7}$$

where $\xi \in \mathcal{X}$, “dist” denotes a Riemannian metric in $\mathcal{X}$ and the function $g : [0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ has a global maximum at 0 and is a strictly decreasing bijection for a neighborhood $V$ of 0.

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We assume throughout this presentation that the following condition holds:

**(R1)** for $u$ sufficiently close to $u_F = \varphi(\zeta)$, the event

$$U(u) := \{x \in \mathcal{X} : \ \varphi(x) > u\} = \{X_0 > u\} \tag{8}$$

corresponds to a topological ball centered at $\zeta$. Moreover, the quantity $\mu(U(u))$ varies continuously, as a function of $u$, in a neighbourhood of $u_F$.

So, if at time $j \in \mathbb{N}$ we have an exceedance of the level $u$ sufficiently large, i.e. $X_j(x) > u$, then we have an entrance of the orbit of $x$ in the ball $U(u)$ at time $j$, i.e. $f^j(x) \in U(u)$.

The behaviour of $1 - F(u)$, as $u \to u_F$, depends on the form of $g^{-1}$.

# Connection between EVL and HTS

Motivated by Collet's work, [C01], we obtained:

**Theorem ([FFT10],[FFT11])**

- *If we have HTS G for balls centred on $\xi \in \mathcal{X}$, then we have an EVL for $M_n$ with $H = G$.*

**Theorem ([FFT10],[FFT11])**

- If we have an EVL $H$ for $M_n$, then we have HTS $G = H$ for balls centred on $\xi$.

Idea of the proof:

$$\{x : M_n(x) \le u_n\} = \bigcap_{j=0}^{n-1} \{x : X_j(x) \le u_n\}$$

$$= \bigcap_{j=0}^{n-1} \{x : g(\mathrm{dist}(f^j(x), \xi)) \le u_n\}$$

$$= \bigcap_{j=0}^{n-1} \{x : \mathrm{dist}(f^j(x), \xi) \ge g^{-1}(u_n)\} = \{x : r_{B_{g^{-1}(u_n)}(\xi)}(x) \ge n\}$$

Thus,

$$\mu\{x : M_n(x) \le u_n\} = \mu\{x : r_{B_{g^{-1}(u_n)}(\xi)}(x) \ge n\}$$

Note that

$$\frac{\tau}{n} \sim 1 - F(u_n) = \mu\left(B_{g^{-1}(u_n)}(\xi)\right) \Leftrightarrow n \sim \frac{\tau}{\mu\left(B_{g^{-1}(u_n)}(\xi)\right)}$$

and so

$$\mu\{x : M_n(x) \leq u_n\} \sim \mu\left\{x : r_{B_{g^{-1}(u_n)}(\xi)}(x) \geq \frac{\tau}{\mu\left(B_{g^{-1}(u_n)}(\xi)\right)}\right\} \to 1 - G(\tau)$$

Consider now a sequence $\delta_n \to 0$. We want to study

$$\mu\left(\left\{x : r_{B_{\delta_n}(\xi)}(x) < \frac{t}{\mu(B_{\delta_n}(\xi))}\right\}\right)$$

Choose $\ell_n$ such that $g^{-1}(u_{\ell_n}) \sim \delta_n$. We have that

$$\{x : M_{\ell_n}(x) \leq u_{\ell_n}\} = \bigcap_{j=0}^{\ell_n - 1} \{x : X_j(x) \leq u_{\ell_n}\}$$

$$= \bigcap_{j=0}^{\ell_n - 1} \{x : g(\text{dist}(f^j(x), \xi)) \leq u_{\ell_n}\}$$

$$= \bigcap_{j=0}^{\ell_n - 1} \{x : \text{dist}(f^j(x), \xi) \geq g^{-1}(u_{\ell_n})\} = \{x : r_{B_{g^{-1}(u_{\ell_n})}(\xi)}(x) \geq \ell_n\}$$

As before,

$$\frac{\tau}{\ell_n} \sim 1 - F(u_{\ell_n}) = \mu\left(B_{\delta_n}(\xi)\right) \sim \mu\left(B_{g^{-1}(u_{\ell_n})}(\xi)\right) \Leftrightarrow \ell_n \sim \frac{\tau}{\mu\left(B_{\delta_n}(\xi)\right)}.$$

In this way,

$$\mu\left\{x : r_{B_{\delta_n}(\xi)}(x) < \frac{\tau}{\mu(B_{\delta_n}(\xi))}\right\} \sim 1 - \mu\{x : M_{\ell_n}(x) \leq u_{\ell_n}\} \to H(\tau)$$

Assuming $D(u_n)$ holds, let $(k_n)_{n\in\mathbb{N}}$ be a sequence of integers such that

$$k_n \to \infty \quad \text{and} \quad k_n t_n = o(n). \tag{9}$$

**Condition ($D'(u_n)$)**

We say that $D'(u_n)$ holds for the sequence $X_0, X_1, \ldots$ if

$$\limsup_{n\to\infty} n \sum_{j=1}^{[n/k]} P\{X_0 > u_n \text{ and } X_j > u_n\} = 0. \tag{10}$$

**Theorem (Leadbetter)**

*Let $\{u_n\}$ be such that $n(1 - F(u_n)) \to \tau$, as $n \to \infty$, for some $\tau \geq 0$. Assume that conditions $D(u_n)$ and $D'(u_n)$ hold. Then*

$$P(M_n \leq u_n) \to e^{-\tau} \quad \text{as } n \to \infty.$$

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$$P(M_n \leq u_n) \to e^{-\tau} \quad \text{as } n \to \infty.$$

Motivated by the work of Collet (2001) we introduced:

**Condition ($D_2(u_n)$)**

We say that $D_2(u_n)$ holds for the sequence $X_0, X_1, \ldots$ if for any integers $\ell, t$ and $n$

$$|P\{X_0 > u_n \cap \max\{X_t, \ldots, X_{t+\ell-1} \leq u_n\}\} - P\{X_0 > u_n\}P\{M_\ell \leq u_n\}| \leq \gamma(n, t),$$

where $\gamma(n, t)$ is nonincreasing in $t$ for each $n$ and $n\gamma(n, t_n) \to 0$ as $n \to \infty$ for some sequence $t_n = o(n)$.

**Theorem ([FF08a])**

*Let $\{u_n\}$ be such that $n(1 - F(u_n)) \to \tau$, as $n \to \infty$, for some $\tau \geq 0$. Assume that conditions $D_2(u_n)$ and $D'(u_n)$ hold. Then*

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$$P(M_n \leq u_n) \to e^{-\tau} \quad \text{as } n \to \infty.$$

# Periodic points

From here on we are going to assume that:

**(R2)** $\zeta \in \mathcal{X}$ is a repelling periodic point of period $p \in \mathbb{N}$. The periodicity of $\zeta$ implies that for all $u$ sufficiently large, $\{X_0 > u\} \cap \{X_p > u\} \neq \emptyset$ and $\{X_0 > u\} \cap \{X_j > u\} = \emptyset$ for all $j = 1, \ldots, p-1$. The fact that $\zeta$ is repelling means that we have backward contraction implying that there exists $0 < \theta < 1$ such that

$$P\left(\{X_0 > u\} \cap \{X_p > u\}\right) \sim (1-\theta)P(X_0 > u),$$

for all $u$ sufficiently large.

Under this assumption, $D'(u_n)$ does not hold since

$$n \sum_{j=1}^{[n/k_n]} P(X_0 > u_n, X_j > u_n) \geq nP(X_0 > u_n, X_p > u_n) \to (1-\theta)\tau$$

Define the event $Q_{p,0}(u) := \{X_0 > u, X_p \leq u\}$.

Observe that for $u$ sufficiently large, $Q_{p,0}(u)$ corresponds to an annulus centred at $\xi$.

Define the events: $Q_{p,i}(u) := \{X_i > u, X_{i+p} \leq u\}$,

$Q^*_{p,i}(u) := \{X_i > u\} \setminus Q_{p,i}(u)$ and $\mathcal{Q}_{p,s,\ell}(u) = \bigcap_{i=s}^{s+\ell-1} Q^c_{p,i}(u)$.

### Theorem ([FFT12])

*Let* $(u_n)_{n\in\mathbb{N}}$ *be such that* $nP(X_0 > u_n) \to \tau$*, for some* $\tau \geq 0$*. Suppose* $X_0, X_1, \ldots$ *is as in* (6) *and (R2) is satisfied for* $p \in \mathbb{N}$ *and* $\theta \in (0,1)$*. Then*

$$\lim_{n\to\infty} P(M_n \leq u_n) = \lim_{n\to\infty} P(\mathcal{Q}_{p,0,n}(u_n)) \tag{11}$$

- First observe that $\{M_n \leq u_n\} \subset \mathcal{Q}_{p,0,n}(u_n)$.
- Moreover, $\mathcal{Q}_{p,0,n}(u_n) \setminus \{M_n \leq u_n\} \subset \bigcup_{i=0}^{n-1} \{X_i > u_n, X_{i+p} > u_n, \ldots, X_{i+s_ip} > u_n\}$, where $s_i = [\frac{n-1-i}{p}]$.
- It follows by (R2) and stationarity that

$$P(\mathcal{Q}_{p,0,n}(u_n) \setminus \{M_n \leq u_n\}) \leq pP\left(X_0 > u_n, X_p > u_n\right)$$
$$= p(1-\theta)\frac{\tau}{n} \xrightarrow[n\to\infty]{} 0.$$

.

**Condition ($D^p(u_n)$)**

We say that $D^p(u_n)$ holds for$X_0, X_1, \dots$ if for any $\ell, t$ and $n$

$$\left|P\left(Q_{p,0}(u_n) \cap \mathcal{Q}_{p,t,\ell}(u_n)\right) - P(Q_{p,0}(u_n))P(\mathcal{Q}_{p,0,\ell}(u_n))\right| \leq \gamma(n,t),$$

where $\gamma(n,t)$ is nonincreasing in $t$ for each $n$ and $n\gamma(n,t_n) \to 0$ as $n \to \infty$ for some sequence $t_n = o(n)$.

Let $(k_n)_{n\in\mathbb{N}}$ be a sequence of integers such that $k_n \to \infty$ and $k_n t_n = o(n)$.

**Condition ($D'_p(u_n)$)**

We say that $D'_p(u_n)$ holds for the sequence $X_0, X_1, X_2, \dots$ if there exists a sequence $\{k_n\}_{n\in\mathbb{N}}$ satisfying (9) and such that

$$\lim_{n\to\infty} n \sum_{j=1}^{[n/k_n]} P(Q_{p,0}(u_n) \cap Q_{p,j}(u_n)) = 0. \tag{12}$$

**Theorem ([FFT12])**

*Let $(u_n)_{n\in\mathbb{N}}$ be such that $nP(X_0 > u_n) \to \tau$, for some $\tau \geq 0$. Suppose $X_0, X_1, \ldots$ is as in (6) and (R1) and (R2) are satisfied. Assume further that conditions $D^p(u_n)$ and $D'_p(u_n)$ hold. Then*

$$\lim_{n\to\infty} P(M_n \leq u_n) = \lim_{n\to\infty} P(\mathcal{Q}_{p,0,n}(u_n)) = e^{-\theta\tau}. \tag{13}$$

Note that

$$\begin{aligned} &P(Q_{p,0}(u)) = P(X_0 > u, X_p \leq u) = \\ &= P(X_0 > u) - P(X_0 > u, X_p > u) = \\ &\sim P(X_0 > u) - (1-\theta)P(X_0 > u) = \theta P(X_0 > u), \end{aligned}$$

and so

$$\theta \sim \frac{P(Q_{p,0}(u))}{P(X_0 > u)}.$$

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## Decay of correlations implies $D_2(u_n)$

Suppose that there exists a nonincreasing function $\gamma : \mathbb{N} \to \mathbb{R}$ such that for all $\phi : \mathcal{X} \to \mathbb{R}$ with bounded variation and $\psi : \mathcal{X} \to \mathbb{R} \in L^\infty$, there is $C > 0$ independent of $\phi, \psi$ and $n$ such that

$$\left| \int \phi \cdot (\psi \circ f^t) d\mu - \int \phi d\mu \int \psi d\mu \right| \leq C \mathrm{Var}(\phi) \|\psi\|_\infty \gamma(t), \quad \forall n \geq 0, \tag{14}$$

where $\mathrm{Var}(\phi)$ denotes the total variation of $\phi$ and $n\gamma(t_n) \to 0$, as $n \to \infty$ for some sequence $t_n = o(n)$.

Taking $\phi = \mathbf{1}_{\{X > u_n\}}$ and $\psi = \mathbf{1}_{\{M_\ell \leq u_n\}}$, then

$$(14) \Rightarrow D_2(u_n),$$

(with $\gamma(n, t) = C\mathrm{Var}(\mathbf{1}_{\{X > u_n\}}) \|\mathbf{1}_{\{M_\ell \leq u_n\}}\|_\infty \gamma(t) \leq C' \gamma(t)$ and for the sequence $\{t_n\}$ such that $t_n/n \to 0$ and $n\gamma(t_n) \to 0$ as $n \to \infty$).

## Decay of correlations against $L^1$ implies $D'_p(u_n)$

Suppose that there exists a nonincreasing function $\gamma : \mathbb{N} \to \mathbb{R}$ such that for all $\phi : \mathcal{X} \to \mathbb{R}$ with bounded variation and $\psi : \mathcal{X} \to \mathbb{R} \in L^1$, there is $C > 0$ independent of $\phi, \psi$ and $n$ such that

$$\left| \int \phi \cdot (\psi \circ f^t) d\mu - \int \phi d\mu \int \psi d\mu \right| \leq C \mathrm{Var}(\phi) \|\psi\|_1 \gamma(t), \quad \forall n \geq 0, \tag{15}$$

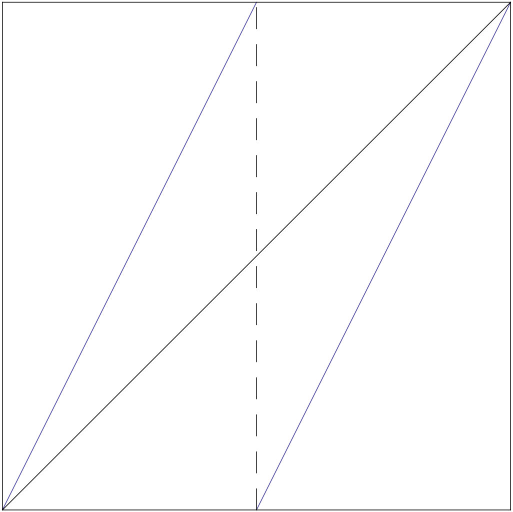
where $\mathrm{Var}(\phi)$ denotes the total variation of $\phi$ and $n\gamma(t_n) \to 0$, as $n \to \infty$ for some sequence $t_n = o(n)$.

Taking $\phi = \mathbf{1}_{Q_p(u_n)}$ and $\psi = \mathbf{1}_{Q_p(u_n)}$, then

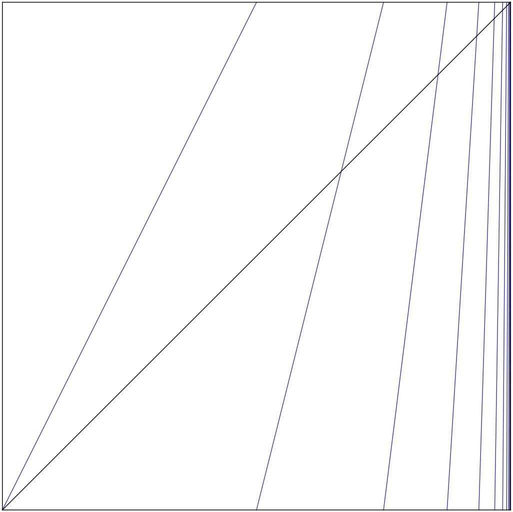
$$(15) \Rightarrow D'_p(u_n),$$

$P(Q_{p,0}(u_n) \cap Q_{p,j}(u_n)) \leq P(Q_{p,0}(u_n))^2 + C' P(Q_{p,0}(u_n))\gamma(j) \lesssim (\tau/n)^2 + C'(\tau/n)\gamma(j)$.

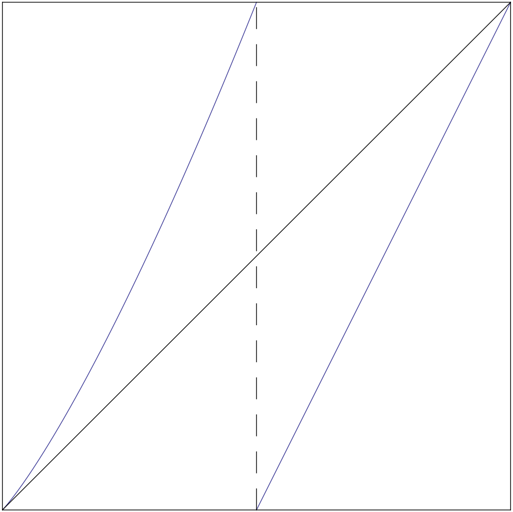
# Doubling map



# Rychlik map



# Intermittent map



# Benedicks-Carleson maps