Central Limit Theorem for dimension of Gibbs measures in hyperbolic dynamics

Renaud Leplaideur& Benoît Saussol *

September 15, 2010

Abstract

For an equilibrium measure of an Hölder potential, we prove an analogue of the Central Limit Theorem for the fluctuations of the logarithm of the measure of balls as the radius goes to zero.

A noticeable consequence is that when this measure is not absolutely continuous, the probability that a ball of radius ε chosen at random have a measure smaller (or larger) than ε^{δ} is asymptotically equal to 1/2, where δ is the Hausdorff dimension of the measure.

Our method applies to a class of non-conformal expanding maps on the d-dimensional torus. It also applies to conformal repellers and Axiom A surface diffeomorphisms and possibly to a class of one-dimensional non uniformly expanding maps. These generalizations are presented at the end of the paper.

Keywords: Gibbs measure, expanding maps, dimension, central limit theorem.
MSC: 37A35, 37C45, 37D35, 60F05.

1 Introduction

1.1 General background and motivations

Let consider a $C^{1+\alpha}$ diffeomorphism or expanding map T acting on some compact Riemaniann manifold X. We can associate to each T-invariant probability μ several global quantities: the Kolmogorov entropy h_μ, the family of Lyapunov exponents $\lambda_{\mu,1} < \lambda_{\mu,2} < \ldots < \lambda_{\mu,k}$ and the Hausdorff dimension δ_μ; the dimension δ_μ being the infimum of all the Hausdorff dimensions of sets with full μ-measure. When the measure is ergodic and hyperbolic, in the sense that no Lyapunov exponent is zero, there is a relation between these three quantities.

*Laboratoire de Mathématiques, UMR 6205, Université de Bretagne Occidentale, 6 rue Victor Le Gorgeu BP 809 F - 29285 BREST Cedex, renaud.leplaideur@univ-brest.fr & benoit.saussol@univ-brest.fr

For the case of one dimensional maps, we recall that the Lyapunov exponent is defined by $\lambda_\mu := \int \log |T'| d\mu$. Then, the relation between these three terms is (see *e.g.* [Led81])

$$h_\mu = \delta_\mu \lambda_\mu.$$

For the higher dimensional case, the relation is (see *e.g.* [LY85a])

$$h_\mu = \sum_i \delta_{\mu,i} \lambda^+_{\mu,i},$$

where $\lambda^+_{\mu,i}$ denotes $\max(0, \lambda_{\mu,i})$. The terms $\delta_{\mu,i}$ may be considered as *intermediate unstable* dimensions and we have $\delta^u_\mu = \sum_{i,\, \lambda_{\mu,i}>0} \delta_{\mu,i}$ (similarly, if T^{-1} exists, we have $\delta^s_\mu = \sum_{i,\, \lambda_i<0} \delta_{\mu,i}$). On the other hand, associated to the measure μ, there is a notion of local (or pointwise) dimension. We set

$$\delta_\mu(x) := \lim_{\varepsilon \to 0} \frac{\log \mu(B(x,\varepsilon))}{\log \varepsilon} \tag{1}$$

whenever the limit exists. Here $B(x,\varepsilon)$ is the open ball of radius ε centered at x. It is known (see [LY85b] and [BPS99]) that for μ-almost every point x, the pointwise dimension $\delta_\mu(x)$ exists, is equal to δ_μ and $\delta_\mu = \delta^u_\mu + \delta^s_\mu$.

In this article we want to focus on the non-conformal case, *i.*e. when there are several Lyapunov exponents (possibly) with the same sign. In that case, the dimension theory is still incomplete: beside the one stated above there are few general results. Most of the finer results (multifractal analysis, existence of measure of maximal dimension, etc.) are established under additional assumptions. Possibly, one explanation is that for the non-conformal dynamics, the dynamical objects as cylinders are far away of being "round" balls. For example, the simple map T defined on $(\mathbb{R}/\mathbb{Z})^2$ by

$$T(x,y) = \begin{pmatrix} 2x & \mod 1 \\ 3y & \mod 1 \end{pmatrix}$$

and some nonlinear generalizations of this non-conformal expanding map are still a subject of investigation [McM84, GP97, Luz06, BF09]. These maps satisfy our assumption.

The convergence in (1), loosely speaking, means that $\mu(B(x,\varepsilon))$ is of the order ε^δ_μ for μ-a.e. point x. In this article we make precise the fluctuations in this convergence. The motivation is to give substance to the approximation $\mu(B(x,\varepsilon)) \approx \varepsilon^{\delta_\mu}$. Namely, we prove a central limit theorem

$$\frac{\log \mu(B(x,\varepsilon)) - \delta_\mu \log \varepsilon}{\sqrt{-\log \varepsilon}} \overset{\mathcal{D}}{\Longrightarrow} \mathcal{N}(0,\sigma^2).$$

A consequence is Corollary 1.2: when $\sigma \neq 0$, then asymptotically as $\varepsilon \to 0$, half of the balls of radius ε have a measure smaller than ε^{δ_μ} and half of them have a measure larger than ε^{δ_μ}; See Corollary 1.2 for the precise statement.

The proof of the central limit theorem requires us to work at the *level of processes*. That is, at some point, we need a *functional* central limit theorem. As a byproduct of our method we also get the functional version of the above central limit theorem, which is the statement of our main theorem that we will now present in detail.

1.2 Statement of the Main Theorem

1.2.1 The dynamics

We consider $T : \mathbb{T}^2 \circlearrowleft$ a $\mathcal{C}^2$ map. We assume that T is (uniformly) expanding, in the sense that

$$\sup_x \|(d_x T)^{-1}\| < 1$$

Consider a Hölder continuous function $\varphi : \mathbb{T}^2 \to \mathbb{R}$ called the *potential*, and define its pressure by

$$P(\varphi) := \sup \left\{ h_\mu + \int \varphi \, d\mu \right\},$$

where the supremum is considered on the set of T-invariant probabilities. In this setting the supremum is attained at a unique invariant measure μ_φ, which is called the equilibrium state of φ (see [Rue04]).

Note that considering such a potential, we can assume that the pressure is equal to zero. This can be realized easily replacing φ by φ minus the pressure.

1.2.2 Skorohod topology

In this article we shall use the Skorohod topology. We refer to [Bil99] chapter 3 for more global setting on this topology. We denote by $\mathcal{D}([0,1])$ the set of *cadlag* functions on $[0,1]$ endowed with the Skorohod topology:

Two functions u and v in $\mathcal{D}([0,1])$ are ρ-close if there exists $\lambda : [0,1] \to [0,1]$ such that

1. $\lambda(0) = 0$ and $\lambda(1) = 1$ and λ is increasing;
2. $\forall\, t \in [0,1]$, $|\lambda(t) - t| \leq \rho$,
3. $\forall\, t$, $|u(\lambda(t)) - v(t)| \leq \rho$.

1.2.3 Main result and corollaries

We state the main result in dimension 2. A generalization with arbitrary dimension d is stated in Subsection 4.1, where we also explain how we have to adapt the proof of the $d = 2$-case to the general case. We set $\pi(x, y) = x$.

Our main theorem is

Main Theorem. *Let $T : \mathbb{T}^2 \circlearrowleft$ be a $\mathcal{C}^2$ expanding map of the form $T(x, y) = (f(x), g(x, y))$. Let φ be a Hölder continuous function from $\mathbb{T}^2$ to $\mathbb{R}$. Let μ_φ be the equilibrium state associated to φ. Let $\delta := \delta_{\mu_\varphi}$ be its Hausdorff dimension. We assume that $\int \log|f' \circ \pi| \, d\mu_\varphi < \int \log\left|\frac{\partial g}{\partial y}\right| d\mu_\varphi$ holds.*

Then, there exists a real number $\sigma \geq 0$ such that the process

$$\frac{\log \mu_\varphi \left(B((x,y),\varepsilon^t)\right) - t\delta \log \varepsilon}{\sqrt{-\log \varepsilon}}, \quad t \in [0,1]$$

converges in $\mathcal{D}([0,1])$ in distribution under the law of μ_φ to the process $\sigma W(t)$, where W is the standard Wiener process.

In addition, the variance σ^2 is zero if and only if μ_φ is the unique absolutely continuous invariant measure, or equivalently φ is cohomologous to $-\log|\det DT|$.

As we said above, the hypothesis $\int \log|f' \circ \pi|\, d\mu_\varphi < \int \log \left|\frac{\partial g}{\partial y}\right| d\mu_\varphi$ means that we are dealing with the non-conformal case (see Lemma 2.2). We emphasize that for the absolutely continuous invariant measure, the measure of balls is completely governed by its density h with respect to the Lebesgue measure: in that case we have $\delta = 2$ and the density is continuous (in fact C^1), therefore we have

$$\lim_{\varepsilon \to 0} \frac{\mu_\varphi(B((x,y),\varepsilon))}{\varepsilon^2} = h(x,y)$$

for any $(x,y) \in \mathbb{T}^2$. Needless to say, there is no point in looking at fluctuations in this case.

We emphasize that the CLT below is the main goal of the paper, but the method, at the level of processes, gives as a byproduct several standard corollaries; we refer to [Bil99] for further precisions about functions of Brownian motion paths.

Corollary 1.1 (Central limit theorem)**.** *With the same assumptions and notations, the family of random variables*

$$\frac{\log \mu_\varphi \left(B((x,y),\varepsilon)\right) - \delta \log \varepsilon}{\sqrt{-\log \varepsilon}}$$

converges in distribution to the (possibly degenerate) gaussian distribution $\mathcal{N}(0,\sigma^2)$.

Namely, if $\sigma \neq 0$ we have for every $a < b$ in $\mathbb{R}$,

$$\lim_{\varepsilon \to 0} \mu_\varphi \left(\left\{ (x,y) \in \mathbb{T}^2 \colon \frac{\log \mu_\varphi \left(B((x,y),\varepsilon)\right) - \delta \log \varepsilon}{\sqrt{-\log \varepsilon}} \in [a,b] \right\} \right) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}u^2} du.$$

Another immediate consequence is the precise balance between “heavy” and “light” balls, already mentioned in the introduction:

Corollary 1.2 (Median)**.** *With the same assumptions and notations, if μ_φ is not absolutely continuous then*

$$\mu_\varphi \left(\left\{ (x,y) \in \mathbb{T}^2 \colon \mu_\varphi \left(B((x,y),\varepsilon)\right) \leq \varepsilon^\delta \right\} \right) \to \frac{1}{2}.$$

Two other consequences follow from [Bil99] (see resp. formulas (9.14) and (9.27)):

Corollary 1.3 (Maximum and minimum)**.** *With the same assumptions and notations, if μ_φ is not absolutely continuous then*

$$\mu_\varphi\left(\left\{(x,y)\in\mathbb{T}^2\colon \forall t\in[0,1], \mu_\varphi\left(B((x,y),\varepsilon^t)\right)\le \varepsilon^{t\delta+b\sigma/\sqrt{-\log\varepsilon}}\right\}\right)\to\mathcal{M}(b),$$

where

$$\mathcal{M}(b)=P(\sup_{t\in[0,1]}W(t)\le b)=1-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{2k+1}e^{-\pi^2(2k+1)^2/8b^2}.$$

Corollary 1.4 (Arcsine law)**.** *With the same assumptions and notations, if μ_φ is not absolutely continuous then, the family of random variables*

$$\mathcal{T}_\varepsilon(x,y):=Leb\left(t\in[0,1]\colon \mu_\varphi\left(B((x,y),\varepsilon^t)\right)\le\varepsilon^{t\delta}\right)$$

converges in distribution to the arcsine law (recall that U follows the arcsine law if $P(U\le u)=\frac{2}{\pi}\arcsin\sqrt{u}$).

1.3 Steps of the proof and structure of the paper

The proof has two main steps. In a first part (Section 2), we use dynamical and ergodic arguments to reduce the problem to the study of the convergence of some process of the form (see Lemma 2.17)

$$\frac{S_{n_{\varepsilon^t}}\phi_1+S_{m_{\varepsilon^t}}\phi_2}{\sqrt{-\log\varepsilon}}, \tag{2}$$

where n_ε and m_ε are random "times".

Then, in Section 3 we use arguments from Probability Theory to prove the convergence of this last process. These arguments are somehow general and independent of the functions ϕ_1 and ϕ_2.

We mention that the use of the Skorohod topology is perhaps not necessary. It seems useful because the process we study is *a priori* discontinuous. However, note that the limit process is *a.e.* continuous. Therefore the convergence is uniform. Nevertheless, the space of cadlag functions endowed with the norm of uniform convergence is not separable, which may cause some troubles as pointed out by P. Billingsley in [Bil99]. We thus preferred to work in $\mathcal{D}([0,1])$.

Our method works for the higher dimension case and also applies to conformal repeller and Axiom A surface diffeomorphisms. These adaptations are presented in Section 4. Hypothesis of uniform expansion does not seem to be so crucial and we also discuss some possible extensions of our main result at the end of the paper.

From now until the end of the paper, we assume that $T(x,y):=(f(x),g(x,y))$ and φ satisfy the assumptions of the Main Theorem.

2 Reduction to a non-homogeneous sum of random variables

2.1 An adapted family of fibered partitions

It is well-known that with our assumption there exists a Markov partition with finitely many proper sets. We prefer here to work with a more naive partition. We emphasize that "partition" is here understood up to a set with null μ_φ-measure.

Given $(x_0, y_0) \in \mathbb{T}^2$ we denote $S_0 = \{x_0\} \times \mathbb{T} \cup \mathbb{T} \times \{y_0\}$. Then, for every $n \geq 0$, let the partition $\mathcal{R}_n$ be the collection of connected components of $T^{-n}(\mathbb{T}^2 \setminus S_0)$. Similarly we define $\mathcal{P}_n$ as the collection of connected components of $f^{-n}(\mathbb{T} \setminus \{x_0\})$.

The main properties of these partitions are listed in the following lemma. Its proof is left to the reader.

Lemma 2.1. *For any $(x_0, y_0) \in \mathbb{T}^2$, there exist a finite partition $\mathcal{R}$ of $\mathbb{T}^2$ in Markov proper sets R_i such that*

1. *Each element R_i of $\mathcal{R}_n$ is a proper set.*
2. *For each element R_i of $\mathcal{R}_n$, $T^n(R_i) = \mathbb{T}^2$ and $T_{|\mathring{R}_i}$ is one-to-one.*
3. *If R_i and R_j are different elements of $\mathcal{R}_n$, then $\pi(\mathring{R}_i) \cap \pi(\mathring{R}_j) = \emptyset$.*
4. *The boundary $\partial\mathcal{R}$ is mapped by T^n to S_0.*
5. *$\mathcal{P}_n = \pi(\mathcal{R}_n)$.*
6. *$T^k(\mathcal{R}_n) = \mathcal{R}_{n-k}$ for $k \leq n$.*
7. *Each element P_i of $\mathcal{R}_n$ is a closed interval.*
8. *For each element P_i of the $\mathcal{P}_n$, $f^n(R_i) = \mathbb{T}$ and $f_{|\mathring{P}_i}$ is one-to-one.*
9. *If P_i and P_j are different elements of $\mathcal{P}_n$, then $\mathring{P}_i \cap \mathring{P}_j = \emptyset$.*
10. *The boundary $\partial\mathcal{P}$ is mapped by f^n to S_0.*
11. *$f^k(\mathcal{P}_n) = \mathcal{P}_{n-k}$ for $k \leq n$.*

The boundary of the partition $\partial\mathcal{R}_n$ is going to play an important role. For a fixed point (x, y) and for an integer n, the boundary or $\mathcal{R}_n(x, y)$ is denoted by $\partial\mathcal{R}_n(x, y)$. It is the union of a vertical boundary $\partial^v\mathcal{R}_n(x, y)$ and a horizontal boundary $\partial^h\mathcal{R}_n(x, y)$. The vertical boundary is exactly the union of two vertical segments (its projection by π is the union of two different points). The horizontal boundary is the union of two "relatively" horizontal curves. Their slope is studied in Lemma 2.3.

2.2 Lyapunov exponents and geometry of the partition

We define for all integer n

$$F_n = \prod_{j=0}^{n-1} f' \circ f^j \circ \pi, \quad G_n = \prod_{j=0}^{n-1} \frac{\partial g}{\partial y} \circ T^j. \tag{3}$$

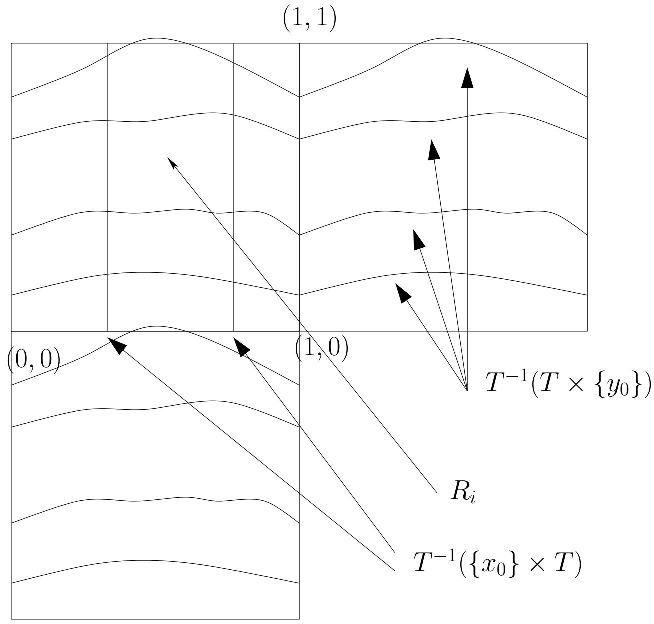

Figure 1: Partition $\mathcal{R}_1$ in nice proper sets

Lemma 2.2. *Let $T : \mathbb{T}^2 \circlearrowleft$ be as in the Theorem. We set $T(x, y) = (f(x), g(x, y))$. There is an invariant splitting $T\mathbb{T}^2 = E^u \oplus E^{uu}$ defined μ-a.e. The two associated Lyapunov exponents of (T, μ) are $\lambda^u := \int \log |f'(x)| \, d\mu_\varphi(x, y)$ and $\lambda^{uu} := \int \log \left| \frac{\partial g}{\partial y}(x, y) \right| \, d\mu_\varphi(x, y)$.*

Proof. By the ergodic theorem we have μ_φ-a.e.

$$\lim \frac{1}{n} \log F_n = \int \log |f' \circ \pi| \, d\mu_\varphi = \lambda^u < \lambda^{uu} = \lim \int \log |\frac{\partial g}{\partial y}| \, d\mu_\varphi = \frac{1}{n} \log G_n. \quad (4)$$

Therefore, the series

$$U = -\sum_{k=0}^{\infty} \frac{F_k}{G_{k+1}} \frac{\partial g}{\partial x} \circ T^k$$

converges almost everywhere. Define the splitting

$$E^u = \begin{pmatrix} 1 \\ U \end{pmatrix}, \quad E^{uu} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

One directly checks that as announced the splitting is invariant:

$$D_{(x,y)}T \begin{pmatrix} 1 \\ U(x, y) \end{pmatrix} = f'(x) \begin{pmatrix} 1 \\ U \circ T(x, y) \end{pmatrix}, \quad D_{(x,y)}T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\partial g}{\partial y}(x, y) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It now remains to prove that λ^u and λ^{uu} are the two Lyapunov exponents of T (for the measure μ_φ). This simply follows from the fact that the two directions E^u

and E^{uu} are DT-invariant and one-dimension vector spaces. The last two equalities show that respective expansions along these directions are exactly λ^u and λ^{uu}. □

We will need some estimates for the top and bottom boundaries $\partial^h\mathcal{R}_n$ of the partition $\mathcal{R}_n$. Note that if a point (x,y) belongs to $\partial^h\mathcal{R}_n$ then, it also belongs to $\partial^h\mathcal{R}_m$ for every $m \geq n$. We denote by $\mathcal{T}_{x,y,n}$ the slope of the tangent to $\partial^h\mathcal{R}_n$ at (x,y).

Lemma 2.3. *For every n and for μ_φ-almost every (x,y) there exists a real number $A_{\partial^h}(x,y)$ such that for every (x',y') in $\partial^h\mathcal{R}_n(x,y)$,*

$$|\mathcal{T}_{x',y',n}| \leq A_{\partial^h}(x,y).$$

Proof. We assume that (x,y) is such that the invariant splitting is defined. For (x',y') in $\partial^h\mathcal{R}_n(x,y)$, we set

$$U_n := -\sum_{k=0}^{n-1} \frac{F_k(x',y')}{G_{k+1}(x',y')}\frac{\partial g}{\partial x} \circ T^k(x',y').$$

Set $(\alpha,\beta) = (D_{(x',y')}T^n)^{-1}(1,0)$. Then (α,β) is tangent to $\partial^h\mathcal{R}_n(x,y)$ at (x',y'). Moreover we get

$$DT^n = \begin{pmatrix} F_n & 0 \\ -G_nU_n & G_n \end{pmatrix}.$$

Therefore the slope of $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ in the canonical basis is

$$\beta/\alpha = U_n.$$

The bounded distortion property shows that there exists a constant A_T such that for all $(x'',y'') \in \mathcal{R}_n(x,y)$ we have

$$\frac{1}{A_T}|U_n(x,y)| \leq |U_n(x'',y'')| \leq A_T|U_n(x,y)|.$$

We use this double inequality for (x',y'). Hence, $|\mathcal{T}_{x',y',n}| \leq A_T|U_n(x,y)|$ holds. Finally $U_n(x,y)$ converges to U for a.e. (x,y). It is thus bounded, and the lemma is proved. □

2.3 Multi-temporal Markov approximation of balls

Definition 2.4. *Let ε be a positive real number.*
(i) We denote by $n_\varepsilon(x,y)$ the largest integer k such that $G_k(x,y)\varepsilon \leq 1$
(ii) we denote by $m_\varepsilon(x)$ the largest integer k such that $F_k(x)\varepsilon \leq 1$.

Lemma 2.5. *There exists some constant $c > 0$ such that $c \leq F_{m_\varepsilon(x)}(x)\varepsilon \leq 1$ and $c \leq G_{n_\varepsilon(x,y)}(x,y)\varepsilon \leq 1$.*

Proof. The inequalities follow directly from the definition and the fact that the functions f' and $\frac{\partial g}{\partial y}$ are bounded from above and from below by a positive constant. □

Lemma 2.6. *For μ_φ a.e. point (x,y) we have* $\lim_{\varepsilon\to 0}\frac{n_\varepsilon(x,y)}{-\log\varepsilon}=\frac{1}{\lambda^{uu}}$ *and* $\lim_{\varepsilon\to 0}\frac{m_\varepsilon(x)}{-\log\varepsilon}=\frac{1}{\lambda^{u}}$*. In particular we have $n_\varepsilon(x,y)\ll m_\varepsilon(x)$ (as $\varepsilon\to 0$) for μ_φ a.e. (x,y).*

Proof. This is an immediate consequence of Equation (4) in the proof of Lemma 2.2 and Lemma 2.5. □

Definition 2.7. *We define the multi-temporal approximation of a ball by*

$$C_\varepsilon(x,y):=\mathcal{R}_{n_\varepsilon(x,y)}(x,y)\cap\pi^{-1}(\mathcal{P}_{m_\varepsilon(x)}(x)).$$

This set is in spirit an approximation of the ball $B((x,y),\varepsilon)$. We shall discuss this fact now.

Lemma 2.8. *Let (x,y) be fixed in $\mathbb{T}^2$. The map $T^{n_\varepsilon(x,y)}$ is one-to-one from $\mathring{\mathcal{R}}_{n_\varepsilon(x,y)}$ $\cap\{x\}\times\mathbb{T}$ to $f^{n_\varepsilon(x,y)}(x)\times(\mathbb{T}\setminus\{y_0\})$.*

Proof. $T^{n_\varepsilon(x,y)}$ is one-to-one from the interior of the cylinder $\mathcal{R}_{n_\varepsilon(x,y)}$ to $\mathbb{T}^2\setminus S_0$ and preserves vertical fibers. □

Lemma 2.9. *There exists a constant $D>0$ such that for μ_φ-a.e. point (x,y)* $\operatorname{diam}\mathcal{P}_{m_\varepsilon(x)}(x)\le D\varepsilon$ *and* $\operatorname{diam}(\mathcal{R}_{n_\varepsilon(x,y)}(x,y)\cap\{x\}\times\mathbb{T})\le D\varepsilon$.

Proof. The first assertion follows immediately from the mean value theorem, bounded distortion property, and the fact that $f^{m_\varepsilon(x)}$ is one-to-one on $\mathring{\mathcal{P}}_{m_\varepsilon(x)}$.

For the second one, a vertical segment based on x and contained in $\mathcal{R}_{n_\varepsilon(x,y)}(x,y)$ is expanded by $T^{n_\varepsilon(x,y)}$ by a factor $G_{n_\varepsilon(x,y)}(x,y')$ by the mean value theorem, for some y' such that (x,y') in $\mathcal{R}_{n_\varepsilon(x,y)}(x,y)$. The conclusion follows by bounded distortion property, Lemmas 2.5 and 2.8. □

Notations. In the rest of the paper we use vocabulary from the Probability Theory. Namely, we consider random constants and/or random processes over the probability space $(\mathbb{T}^2,\mathcal{B},\mu_\varphi)$, where $\mathcal{B}$ is the Borel σ-field. The random part depends on the point (x,y) chosen in $\mathbb{T}^2$ with respect to the law μ_φ. Constants are constant with respect to the parameter ε. Processes are functions in $t\in[0,1]$. Therefore, we now may forget to specify and note the randomness in x and (x,y). Namely, in the rest of the paper, we may write *e*.g. n_ε and m_ε instead of $n_\varepsilon(x,y)$ and $m_\varepsilon(x)$.

Lemma 2.10. *There is a choice of $(x_0,y_0)\in\mathbb{T}^2$ such that the following holds:*

There exists a constant $\underline{c}<1$, positive almost everywhere, and a function $\overline{c}_\varepsilon>1$, satisfying $\overline{c}_\varepsilon=O(|\log\varepsilon|)$ in an neighborhood of $\varepsilon=0$ and μ_φ-almost everywhere, such that for any $\varepsilon>0$,

$$C_{\underline{c}\varepsilon}(x,y)\subset B((x,y),\varepsilon)\subset C_{\overline{c}_\varepsilon(x,y)\varepsilon}(x,y).$$

Proof. Let $(x', y') \in C_\varepsilon(x, y)$. By the first assertion of Lemma 2.9 we have $d(x, x') \leq D\varepsilon$.

It follows immediately the second assertion of Lemma 2.9 and Lemma 2.3 that $C_\varepsilon(x, y)$ is included in a "bow tie" of vertical size less than $D\varepsilon + 2A_{\partial^h}(x, y)D\varepsilon$ (see Figure 2). Hence for any $\varepsilon > 0$ we have

$$C_\varepsilon(x, y) \subset B((x, y), 2D(1 + A_{\partial^h}(x, y))\varepsilon).$$

Set $\underline{c} := \dfrac{1}{2D(1 + A_{\partial^h}(x, y))}$. We have just proved that $C_{\underline{c}\varepsilon}(x, y) \subset B((x, y), \varepsilon)$ holds.

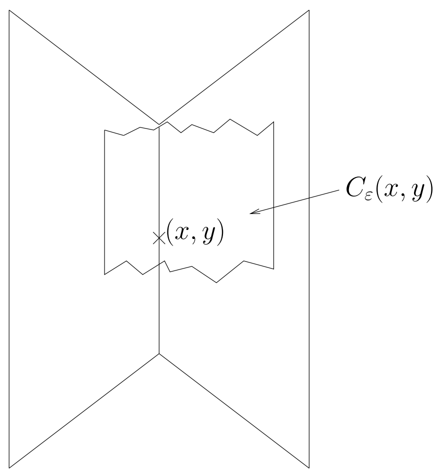

Figure 2: The Markov approximation of the ball contained inside a Bow tie

To get the other inclusion we need to control the distance between a point (x, y) and the boundary of $C_\varepsilon(x, y)$.

We claim that it is possible to choose x_0 and y_0 such that

$$\mu_\varphi(B(\partial\mathcal{R}, r)) \leq ar, \quad \forall r > 0$$

where $a = 8\|DT\|_\infty$.

Indeed, since μ_φ is a probability measure, there exist x_0 and y_0 such that $\mu_\varphi(B(x_0, r) \times \mathbb{T}) \leq 4r$ and for all r, $\mu_\varphi(\mathbb{T} \times B(y_0, r)) \leq 4r$ (see [Sau06], proof of Lemma 3 for details).

We have $B(\partial\mathcal{R}, r) = B(T^{-1}S_0, r) \subset T^{-1}B(S_0, \|DT\|_\infty r)$. Hence by invariance of the measure we get $\mu_\varphi(B(\partial\mathcal{R}, r)) \leq \mu_\varphi(B(S_0, \|DT\|_\infty r)) \leq ar$.

Now, we show that for μ_φ-almost every point the orbit does not approach the boundary $\partial\mathcal{R}$ too "quickly" .

By Borel Cantelli Lemma and the invariance of μ_φ the claim implies that there exists $N = N(x, y)$, finite a.e., such that for any $n \geq N$ we have $d(T^n(x, y), \partial\mathcal{R}) > 1/n^2$. In addition, the distance $d_N(x, y) := d((x, y), \partial\mathcal{R}_N)$ is a.e. non zero since $\cup_{n=0}^N T^{-n} S_0$ has zero measure.

Note that $DT^n = \begin{pmatrix} F_n & 0 \\ -G_nU_n & G_n \end{pmatrix}$. Hence for μ_φ-a.e. (x,y) we have

$$\sup_{\mathcal{R}_n(x,y)} |DT^n| \leq \kappa(x,y)|G_n(x,y)|$$

for some constant $\kappa > 1$ finite a.e..

Let $\rho_n = \dfrac{1}{n^2\kappa|G_n|}$. Let n so large that $\rho_n < d_N(x,y)$. By induction we have $B((x,y),\rho_n) \subset \mathcal{R}_n(x,y)$. Indeed, suppose that for some $N \leq k \leq n-1$ we have $B((x,y),\rho_n) \subset \mathcal{R}_k(x,y)$. Since the image $T^kB((x,y),\rho_n)$ is contained in the ball $B((x,y),\kappa|G_k|\rho_n)$, which does not intersect the boundary $\partial\mathcal{R}$, we get $B((x,y),\rho_n) \subset \mathcal{R}_{k+1}(x,y)$.

Taking $n = n_\varepsilon$ (when ε is sufficiently small) we get

$$B((x,y),\rho_{n_\varepsilon}) \subset \mathcal{R}_{n_\varepsilon}(x,y).$$

A similar and easier argument applied to the one-dimensional map f and the partition $\mathcal{P}$ gives that for some sequence, say, $\rho'_m = \dfrac{1}{m^2\kappa'|F_m|}$ we have

$$B(x,\rho'_{m_\varepsilon}) \subset \mathcal{P}_{m_\varepsilon}(x).$$

Putting together these two inclusions, for any $\varepsilon > 0$ sufficiently small we get that

$$B((x,y),\min(\rho_{n_\varepsilon},\rho'_{m_\varepsilon})) \subset C_\varepsilon(x,y). \tag{5}$$

To get the last inclusion, we rewrite (5) with a variable α instead of ε:

$$B((x,y),\min(\rho_{n_\alpha},\rho'_{m_\alpha})) \subset C_\alpha(x,y).$$

Now, we want to inverse the expression in α and ε: for a given ε, there is α such $\min(\rho_{n_\alpha},\rho'_{m_\alpha}) = \varepsilon$. Hence

$$B((x,y),\varepsilon) \subset C_{\overline{c}_\varepsilon.\varepsilon}(x,y)$$

holds if we set $\overline{c}_\varepsilon = \dfrac{\alpha}{\varepsilon}$.

Note that we can always assume that the constant κ and κ' are bigger than 1. Hence, Lemma 2.5 yields that α is (much) bigger than ε. This shows that $n_\varepsilon(x,y)$ and $m_\varepsilon(x)$ are respectively bigger than $n_\alpha(x,y)$ and $m_\alpha(x)$.

Assuming, for instance, that $\rho_\alpha = \varepsilon$, we get

$$\overline{c}_\varepsilon = n_\alpha^2\kappa|G_{m_\alpha}|\alpha.$$

Again, we use Lemma 2.5, and then Lemma 2.6 to get

$$\overline{c}_\varepsilon \leq \widetilde{\kappa}(x,y)|\log\varepsilon|,$$

for some constant $\widetilde{\kappa}$ *a.e.* finite. □

Remark 1. A direct consequence of Lemma 2.10 is that $\dfrac{\log\overline{c}_\varepsilon}{|\log^{\frac{1}{4}}\varepsilon|}$ is bounded from above when ε belongs to $(0,\frac{1}{2}]$.

2.4 The projected measure ν_φ is a Gibbs measure

We define the projected measure $\nu_\varphi = \pi_* \mu_\varphi$ on $\mathbb{T}$ by

$$\nu_\varphi(A) := \mu_\varphi(A \times \mathbb{T}).$$

As T is a fibred map on $\mathbb{T}^2$ the measure ν_φ is f-invariant. The goal of this subsection is to prove that ν_φ is a Gibbs measure.

This comes from [CU09] (see also [KP09] for alternative proof):

Definition 2.11 (Amalgamation map)**.** *Let A, B be two finite alphabets, with $Card(A) > Card(B)$, and $\pi : A \to B$ be a surjective map (amalgamation) which extends to the map $\pi : A^{\mathbb{N}} \to B^{\mathbb{N}}$ (we use the same letter for both) such that $(\pi \mathbf{a})_n = \pi(\mathbf{a}_n)$ for all $n \in \mathbb{N}$. The map π is continuous and shift-commuting, i.e. it is a factor map from $A^{\mathbb{N}}$ onto $B^{\mathbb{N}}$.*

We remind that the the variation is $\operatorname{var}_n \phi = \sup_C \sup_{x,y \in C} |\phi(x) - \phi(y)|$ where the supremum is taken among all the cylinders C of rank n.

Theorem 2.1 (Chazottes-Ugalde)**.** *Let $\pi : A^{\mathbb{N}} \to B^{\mathbb{N}}$ be the amalgamation map just defined and $\varphi : A^{\mathbb{N}} \to \mathbb{R}$ be a potential with exponentially decaying variation: $\operatorname{var}_n(\varphi) \in O(e^{-qn})$, for some $q > 0$. Then the measure $\mu_\varphi \circ \pi^{-1}$ is a Gibbs measure with support $B^{\mathbb{N}}$, for a potential $\psi : B^{\mathbb{N}} \to \mathbb{R}$ with stretched exponential variation: $\operatorname{var}_n(\psi) \in O(e^{-c\sqrt{n}})$ for some $c > 0$.*

Using our vocabulary and our notation we get:

Proposition 2.12. *There exists a function ψ which satisfies*
(i) the variation of ψ is stretched exponential.
(ii) the measure ν_φ is a Gibbs measure for $(\mathbb{T}, f)$ associated to the potential ψ.

Remark 2. Without loss of generality we set the pressure of ψ with respect to $(\mathbb{T}, f)$ to zero. In particular we have $h_{\nu_\varphi}(f) = -\int \psi \circ \pi d\mu_\varphi$.

2.5 The measure of balls as Birkhoff sums

Notation. For two random variables a_ε and b_ε we use the notation $a_\varepsilon \approx b_\varepsilon$ to mean that there exists $\widetilde{\varepsilon} > 0$ such that for μ_φ-a.e. (x, y),

$$\sup_{0<\varepsilon<\widetilde{\varepsilon}} |a_\varepsilon(x,y) - b_\varepsilon(x,y)| < +\infty$$

Let us recall the definition of the main process

$$N_\varepsilon(t) = \frac{\log \mu_\varphi(B((x,y), \varepsilon^t)) - t\delta \log \varepsilon}{\sqrt{-\log \varepsilon}}, \quad t \in [0,1].$$

By regularity of the measure, N_ε is cadlag[1]. We want to show the convergence of N_ε for the Skorohod topology on $[0,1]$.

[1]Presumably $N_\varepsilon(t)$ is even continuous. However, the proof of that fact would need more space than the margin allows us.

Now, we define another process

$$N'_\varepsilon(t) = \frac{\log \mu_\varphi(C_{\varepsilon^t}(x,y)) - t\delta \log \varepsilon}{\sqrt{-\log \varepsilon}}, \quad t \in [0,1].$$

Lemma 2.13. *If the process N'_ε converges in distribution on $\mathcal{D}([0,1])$ to a Wiener process of variance σ^2 then N_ε converges in distribution to the same process.*

Proof. Observe that the process N_ε has the scale invariance

$$N_\varepsilon(t) = \sqrt{2} N_{\varepsilon^2}(t/2), \quad \forall t \in [0,1].$$

Since the Wiener process itself has the same scale invariance, and the mapping $w(\cdot) \mapsto \sqrt{2}w(\cdot/2)$ is continuous, it is sufficient to prove the convergence in distribution of the process N_ε on $\mathcal{D}([0,1/2])$.

Let $\underline{c}$ and $\overline{c}_\varepsilon$ given by Lemma 2.10. For any $\varepsilon < 1/e^4$, on the set $\Omega^0_\varepsilon := \{\log \underline{c} \geq -\log^{1/4} \frac{1}{\varepsilon}\}$ and for any $t \leq 1/2$ we have

$$\begin{aligned} N_\varepsilon(t) &\geq \frac{\log \mu_\varphi(C_{\underline{c}\varepsilon^t}(x,y)) - t\delta \log \varepsilon}{\sqrt{-\log \varepsilon}} \\ &\geq \frac{\log \mu_\varphi(C_{\exp(-\log^{1/4} \frac{1}{\varepsilon})\varepsilon^t}(x,y)) - t\delta \log \varepsilon}{\sqrt{-\log \varepsilon}} \\ &\geq N'_\varepsilon(t + \log^{-3/4} \frac{1}{\varepsilon}) - \delta \log^{-1/4} \frac{1}{\varepsilon} =: U_\varepsilon(t) \end{aligned}$$

since $\exp(-\log^{1/4} \frac{1}{\varepsilon}) = \varepsilon^{\log^{-3/4} \frac{1}{\varepsilon}}$.

On the other hand, on the set $\Omega^1_\varepsilon = \{\log \overline{c}_\eta \leq \log^{1/8} \frac{1}{\varepsilon} \log^{1/4} \frac{1}{\eta}, \forall \eta \in (0, \frac{1}{2})\}$, and for any $t \in [\log^{-5/8} \frac{1}{\varepsilon}, 1/2]$ we have

$$\begin{aligned} N_\varepsilon(t) &\leq \frac{\log \mu_\varphi(C_{\overline{c}_{\varepsilon^t}\varepsilon^t}(x,y)) - \delta \log \varepsilon}{\sqrt{-\log \varepsilon}} \\ &\leq N'_\varepsilon(t - \log^{-5/8} \frac{1}{\varepsilon}) + \delta \log^{-1/8} \frac{1}{\varepsilon} \end{aligned}$$

since[2] $\overline{c}_{\varepsilon^t} \leq \exp(\log^{1/8} \frac{1}{\varepsilon} \log^{1/4} \frac{1}{\varepsilon^t}) \leq \varepsilon^{-\log^{-5/8} \frac{1}{\varepsilon}}$. Note in addition that for $t \in [0, \log^{-5/8} \frac{1}{\varepsilon})$, since μ is a probability measure, it trivially holds the upper bound

$$N_\varepsilon(t) \leq \frac{0 - t\delta \log \varepsilon}{\sqrt{-\log \varepsilon}} \leq \delta \log^{-1/8} \frac{1}{\varepsilon}.$$

Define

$$V_\varepsilon(t) := \delta \log^{-1/8} \frac{1}{\varepsilon} + \begin{cases} N'_\varepsilon(t - \log^{-5/8} \frac{1}{\varepsilon}) & \text{if } t \geq \log^{-5/8} \frac{1}{\varepsilon} \\ 0 & \text{otherwise} \end{cases}.$$

For any $\varepsilon < 1/e^2$, on $\Omega^0_\varepsilon \cap \Omega^1_\varepsilon$ we have the bound on $[0,1/2]$:

$$U_\varepsilon \leq N_\varepsilon \leq V_\varepsilon.$$

[2] For $\varepsilon < e^{-4}$ and for $t > \log^{-5/8} \frac{1}{\varepsilon}$, $\varepsilon^t \leq e^{-4^{3/8}} = 0.186.. < \frac{1}{2}$.

The measure of $\Omega_\varepsilon^0 \cap \Omega_\varepsilon^1$ goes to 1 (see Remark 1 page 11), and both U_ε and V_ε converge in distribution to the same process. We can now conclude the proof[3]:

Denote, for any $q > 0$, the oscillation of a function $w \in \mathcal{D}([0,1])$ by $v(w,q) = \sup_{|t-s|<q} |w(t) - w(s)|$. We have

$$Z_\varepsilon := V_\varepsilon - U_\varepsilon \leq v(N'_\varepsilon, 2\log^{-5/8}\frac{1}{\varepsilon}) + 2\delta \log^{-1/8}\frac{1}{\varepsilon}.$$

Since N'_ε converges in distribution to a Wiener process W, which is continuous, we claim that the oscillation $v(N'_\varepsilon, 2\log^{-5/8}\frac{1}{\varepsilon})$ converges to zero in probability:

let $r > 0$. Since W is almost surely uniformly continuous, there exists $q > 0$ such that $P(v(W,3q) > r/3) < r$. Let $A(q,r) = \{w \in \mathcal{D} \colon v(w,q) > r\}$. The closure of $A(q,r)$ in the Skorohod topology is trivially contained in $A(3q, r/3)$. Moreover, the weak convergence of the measures $P_{N'_\varepsilon}$ to P_W implies

$$\limsup_{\varepsilon \to 0} P_{N'_\varepsilon}(A(q,r)) \leq P_W(A(3q, r/3)) \leq r.$$

Therefore, there exists ε_0 such that for any $\varepsilon < \varepsilon_0$ we have $P(v(N'_\varepsilon, q) > r) \leq r + r$. Let $\varepsilon_1 < \varepsilon_0$ such that $2\log^{-5/8}\frac{1}{\varepsilon_1} < q$. For any $\varepsilon < \varepsilon_1$ we have

$$P(v(N'_\varepsilon, 2\log^{-5/8}\frac{1}{\varepsilon}) > r) \leq 2r.$$

This proves the convergence in probability.

By Slutsky theorem, N_ε also converges in distribution to the Wiener process.

□

Therefore it suffices to show the convergence in distribution of the process $(N'_\varepsilon(t))_{t\in[0,1]}$. The key lemma below relates the measure of the multi-temporal Markov approximation of the ball with a non-homogeneous Birkhoff sum. This is where we use the skew product structure and the Gibbs property of the measure and its projection.

Lemma 2.14. *We have*

$$\log \mu_\varphi(C_\varepsilon(x,y)) \approx S_{n_\varepsilon}(\varphi - \psi \circ \pi)(x,y) + S_{m_\varepsilon}(\psi \circ \pi)(x,y)$$

Proof. Remind that $C_\varepsilon(x,y) := \mathcal{R}_{n_\varepsilon}(x,y) \cap \pi^{-1}(\mathcal{P}_{m_\varepsilon}(x))$. Given $\varepsilon_0 > 0$, set $\Omega(\varepsilon_0) := \{(x,y) \in \mathbb{T}^2 \colon \forall \varepsilon \leq \varepsilon_0, m_\varepsilon(x) \geq n_\varepsilon(x,y)\}$. Since μ_φ is $\exp(-\varphi)$ conformal and T^{n_ε} is 1-1 on C_ε we have

$$\mu_\varphi(T^{n_\varepsilon} C_\varepsilon) = \int_{C_\varepsilon} e^{-S_{n_\varepsilon}\varphi} d\mu_\varphi.$$

Since C_ε is contained in the cylinder $\mathcal{R}_{n_\varepsilon}$, the bounded distortion property gives

$$\log \mu_\varphi(C_\varepsilon) \approx S_{n_\varepsilon}\varphi(x,y) + \log \mu_\varphi(T^{n_\varepsilon} C_\varepsilon)$$

[3]The conclusion could follow from the sandwich theorem. However, a version for processes is not widely known, therefore we prove it directly in our case.

on C_ε. Moreover, Lemma 2.8 gives that $T^{n_\varepsilon}C_\varepsilon = T^{n_\varepsilon}(\mathcal{R}_{n_\varepsilon} \cap \pi^{-1}(\mathcal{P}_{m_\varepsilon})) = \pi^{-1}(f^{n_\varepsilon}\mathcal{P}_{m_\varepsilon})$ and by the Markov property of $(f, \mathcal{P})$ we get $f^{n_\varepsilon}\mathcal{P}_{m_\varepsilon}(x) = \mathcal{P}_{m_\varepsilon - n_\varepsilon}(f^{n_\varepsilon}(x))$. Therefore

$$\log \mu_\varphi(T^{n_\varepsilon}C_\varepsilon) = \log \nu_\varphi(\mathcal{P}_{m_\varepsilon - n_\varepsilon}(f^{n_\varepsilon}(x))) \approx S_{m_\varepsilon - n_\varepsilon}\psi \circ f^{n_\varepsilon}(x)$$

by the Gibbs property of ν_φ. We end up with

$$\log \mu_\varphi(C_\varepsilon) \approx S_{n_\varepsilon}\varphi + S_{m_\varepsilon - n_\varepsilon}\psi \circ \pi \circ T^{n_\varepsilon} = S_{n_\varepsilon}(\varphi - \psi \circ \pi) + S_{m_\varepsilon}\psi \circ \pi.$$

This holds on $\Omega(\varepsilon_0)$. The conclusion follows since $\mu_\varphi(\Omega(\varepsilon_0)) \to 1$ as $\varepsilon_0 \to 0$ by Lemma 2.6.

□

Denote the intermediate entropies by $h^{uu} = h_{\mu_\varphi}(T) - h_{\nu_\varphi}(f)$ and $h^u = h_{\nu_\varphi}(f)$. Since the pressures of (T, φ) and (f, ψ) are zero we get (see Remark 2 page 12) that

$$h^u = -\int \psi \circ \pi d\mu_\varphi, \quad h^{uu} = -\int (\varphi - \psi \circ \pi) d\mu_\varphi. \tag{6}$$

Lemma 2.15. *With the previous notation, we get the next formula for the pointwise dimension:*

$$\frac{h^{uu}}{\lambda^{uu}} + \frac{h^u}{\lambda^u} = \delta.$$

Proof. It follows from Lemmas 2.6 and 2.14 that μ_φ-a.e.

$$\begin{aligned}\lim_{\varepsilon \to 0} \frac{\log \mu_\varphi(C_\varepsilon)}{\log \varepsilon} &= \lim_{\varepsilon \to 0} \frac{n_\varepsilon}{\log \varepsilon} \frac{1}{n_\varepsilon} S_{n_\varepsilon}(\varphi - \psi \circ \pi) + \frac{m_\varepsilon}{\log \varepsilon} \frac{1}{m_\varepsilon} S_{m_\varepsilon}(\psi \circ \pi) \\ &= -\frac{1}{\lambda^{uu}} \int (\varphi - \psi \circ \pi) d\mu_\varphi - \frac{1}{\lambda^u} \int \psi \circ \pi d\mu_\varphi.\end{aligned}$$

Here, we recover that the pointwise dimension of the measure μ_φ exists μ_φ-a.e. and is constant. This together with Equation (6) prove the first equality. Since it is constant, it is necessarily the Hausdorff dimension δ of the measure μ_φ. □

Set $\delta^{uu} = \frac{h^{uu}}{\lambda^{uu}}$, $\delta^u = \frac{h^u}{\lambda^u}$ and define the functions

$$\phi_1 = \varphi - \psi \circ \pi + \delta^{uu} \log \frac{\partial g}{\partial y}, \quad \phi_2 = \psi \circ \pi + \delta^u \log f' \circ \pi. \tag{7}$$

By Equation (6) and Lemma 2.2 we have

$$\int \phi_1 d\mu_\varphi = \int \phi_2 d\mu_\varphi = 0.$$

Proposition 2.16. *If the functions ϕ_1 and ϕ_2 are both cohomologous to zero then φ is cohomologous to $-\log|\det DT|$, and reciprocally.*

Proof. Suppose that ϕ_1 and ϕ_2 are cohomologous to zero.

Since ϕ_2 is T-cohomologous to zero, $\psi - \delta^u \log|f'|$ is f-cohomologous to zero, hence ψ is f-cohomologous to $-\delta^u \log|f'|$. Therefore the f-pressure of $-\delta^u \log|f'|$ is zero. Since f is uniformly expanding this implies that $\delta^u = 1$.

We have that ϕ_1 is cohomologous to $-\log|f'| - \delta^{uu}\log\left|\frac{\partial g}{\partial y}\right|$. Since $\det DT = f'\circ\pi\cdot\frac{\partial g}{\partial y}$ we get that φ is cohomologous to $-\log|\det DT| + (1-\delta^{uu})\log\left|\frac{\partial g}{\partial y}\right|$. But the convexity of the pressure gives

$$0 = P_T(\varphi) \geq P_T(-\log|\det DT|) + (1-\delta^{uu})\int \log\left|\frac{\partial g}{\partial y}\right| d\mu_\varphi = (1-\delta^{uu})\lambda^{uu}.$$

Therefore $\delta^{uu} \geq 1$. On the other hand, $\delta^u + \delta^{uu} = \delta \leq 2$, which implies that $\delta^{uu} = 1$ also, proving the result.

The reciprocal is immediate. □

Define the process

$$N''_\varepsilon(t) := \frac{S_{n_\varepsilon t}\phi_1 + S_{m_\varepsilon t}\phi_2}{\sqrt{-\log\varepsilon}}, \quad t \in [0,1].$$

We are now able to relate the convergence of the two processes.

Lemma 2.17. *There exists a constant $C_0 < +\infty$ a.s. such that*

$$\sup_{t\in[0,1]} \left|N'_\varepsilon(t) - N''_\varepsilon(t)\right| \leq \frac{C_0}{\sqrt{-\log\varepsilon}}$$

for any $\varepsilon > 0$.

Proof. By Lemma 2.15 we have $\delta = \delta^{uu} + \delta^u$, thus by Lemma 2.5 we have

$$-\delta\log\varepsilon \approx \delta^u \log F_{m_\varepsilon} + \delta^{uu}\log G_{n_\varepsilon}.$$

This relation, together with the facts that $\log F_{m_\varepsilon} = S_{m_\varepsilon}\log f'\circ\pi$ and $\log G_{n_\varepsilon} = S_{n_\varepsilon}\log\frac{\partial g}{\partial y}$, and Lemma 2.14 yield

$$\begin{aligned}\log\mu_\varphi(C_\varepsilon(x,y)) - \delta\log\varepsilon &\approx S_{n_\varepsilon}(\varphi - \psi\circ\pi + \delta^{uu}\log\frac{\partial g}{\partial y}) + S_{m_\varepsilon}(\psi\circ\pi + \delta^u\log f'\circ\pi)\\ &= S_{n_\varepsilon}\phi_1 + S_{m_\varepsilon}\phi_2.\end{aligned}$$

Therefore, there exists a constant C_0 finite a.e. on $\mathbb{T}^2$ such that for any ε and $t \in [0,1]$, we have

$$|N'_\varepsilon(t) - N''_\varepsilon(t)| \leq \frac{C_0}{\sqrt{-\log\varepsilon}}.$$

□

To complete the proof of the main theorem we are left to prove the convergence of the process N''_ε toward a (possibly degenerate) Wiener process. Since ϕ_1 and ϕ_2 have a good regularity and are centered it is well known that their Birkhoff sums follow a central limit theorem. However a problem arise here. The “times” n_ε and m_ε are not constant but they depend on the point.

3 Invariance principle, random change of time

The invariance principle consists in an approximation of all the trajectory of the processes $(S_n\phi_1)$ and $(S_m\phi_2)$ by a Brownian motion, and this is what we need in a first step. Then, a random change of time in the process will give us back N''_ε. Observe that it is sufficient to show the convergence in distribution along the subsequence $\varepsilon = e^{-k}$, that is the convergence of the process $\mathcal{X}_k = N''_{e^{-k}}$ in the Skorohod topology.

3.1 Invariance principle

Let $\phi\colon \mathbb{T}^2 \to \mathbb{R}^2$ defined by $\phi = (\phi_1, \phi_2)$. The function ϕ has stretched exponential decay of the variation $\mathrm{var}_n\,\phi$. Hence, if we set $S_n\phi = (S_n\phi_1, S_n\phi_2)$, the central limit theorem holds for $S_n\phi$. Denote by Q the limiting covariance matrix of $\frac{1}{\sqrt{n}}S_n\phi$. Define the process $\mathcal{Y}_k$ by

$$\mathcal{Y}_k(t) = \frac{1}{\sqrt{k}}\left(S_{\lfloor kt\rfloor}\phi + (kt - \lfloor kt\rfloor)\phi\circ T^{\lfloor kt\rfloor}\right).$$

We denote by $\mathcal{C}$ the space $C([0,1],\mathbb{R})$ endowed with the topology of uniform convergence.

The *weak invariance principle for vector* below, or vectorial functional central limit theorem, is a Folklore theorem in this setting. For the sake of completeness, we point out that it is an immediate consequence of the *almost sure invariance principle for vector* valued observables [MN09].

Theorem 3.1. *The process $\mathcal{Y}_k$ converges in distribution in $\mathcal{C}^2$ to a two-dimensional Brownian motion $\mathcal{B} = (\mathcal{B}_t)_{t\in[0,1]}$ with covariance matrix Q.*

Note that $\mathcal{B}$ (and also $\mathcal{Y}_k$) is continuous, hence the Skorohod topology coincides with the topology of uniform convergence.

Writing $Q = U\Lambda U^*$ for some orthogonal matrix U and $\Lambda = \mathrm{diag}(\sigma_1^2, \sigma_2^2)$, we have that $\mathcal{W} := U^*\mathcal{B} = (\sigma_1 W_1, \sigma_2 W_2)$, where W_1 and W_2 are two independent standard Wiener processes.

3.2 Random change of time and conclusion

If n_ε and m_ε were independent and independent of the process $(\mathcal{Y}_k)$ then we could conclude by direct computation, but these independencies are generally false. The good strategy is to make a random change of time in this process. We follow the general line of Billingsley ([Bil99], Theorem 14.4). The setting here is a bit different: two dimensional time, no need for Skorohod topology.

3.2.1 Existence of the limiting distribution.

We recall that λ^{uu} and λ^u are the two Lyapunov exponents of T. Fix $a > 1/\lambda^u$. Let $\mathcal{Y}_{k,i}$, $i = 1, 2$, be the coordinate processes of $\mathcal{Y}_k$. Let $\mathcal{Z}_k$ be the process in

$C(|0,a]^2, \mathbb{R}^2)$ defined by

$$\mathcal{Z}_k(t_1,t_2) = (\mathcal{Y}_{k,1}(t_1), \mathcal{Y}_{k,2}(t_2))$$

for any $(t_1,t_2) \in [0,a]^2$. Let $\tilde{\nu}_k(t) = (n_{e^{-kt}}, m_{e^{-kt}})$. The real functions $\tilde{\nu}_{k,i}(t)$, $i = 1,2$, are not continuous in t. We define $\nu_{k,i}(t)$ as the continuous function obtained from $\tilde{\nu}_{k,i}(t)$ by linear interpolation at the jump points. Namely, $\nu_{k,i}$ is continuous, affine by part, and coincides with $\tilde{\nu}_{k,i}$ at the jump points.

Let $\theta_1 = 1/\lambda^{uu}$, $\theta_2 = 1/\lambda^u$ and define the random variable Φ_k taking values in $C([0,1],[0,a]^2)$ by

$$\Phi_k(t) = \begin{cases} (\nu_{k,1}(t)/k, \nu_{k,2}(t)/k) & \text{if } \nu_{k,1}(1)/k \le a \text{ and } \nu_{k,2}(1)/k \le a \\ (\theta_1 t, \theta_2 t) & \text{otherwise} \end{cases}$$

Let $\gamma\colon C([0,1],\mathbb{R}^2) \to C([0,1],\mathbb{R})$ defined by $\gamma(u)(t) = u_1(t) + u_2(t)$. Note that

$$\mathcal{X}_k = \gamma(\mathcal{Z}_k \circ \Phi_k) + O(\frac{1}{\sqrt{k}}),$$

whenever the condition in the definition of Φ_k holds (both times are less than a), which happens eventually almost surely.

Lemma 3.1. *The processes $(\frac{n_\varepsilon t_1}{-\log\varepsilon})_{t_1\in[0,1]}$ and $((\frac{m_\varepsilon t_2}{-\log\varepsilon})_{t_2\in[0,1]}$ converge in probability, under the law of μ_φ and in $\mathcal{C}$, respectively, to $(\frac{t_1}{\lambda^{uu}})$ and $(\frac{t_2}{\lambda^u})$.*

Proof. By Lemma 2.6, almost everywhere, for any $t_1 \in [0,1]$, $\frac{n_\varepsilon t_1}{-\log\varepsilon}$ converges to $(\frac{t_1}{\lambda^{uu}})$. Since the process is positive and nondecreasing in t_1, it follows from Dini's (or Pólya's) theorem that the convergence is uniform. Hence the process converges almost surely in $\mathcal{C}$, hence in probability. The same is true for m_ε. □

By Lemma 3.1 the map Φ_k converges almost surely in uniform norm to a deterministic map Φ, defined by $\Phi(t) = (\theta_1 t, \theta_2 t)$ for any $t \in [0,1]$.

Define the continuous mapping h from $C([0,1],\mathbb{R}^2)$ to $C([0,1]^2,\mathbb{R}^2)$ by

$$h(y)(t_1,t_2) = (y_1(t_1), y_2(t_2)), \quad y \in C([0,a],\mathbb{R}^2).$$

Lemma 3.2. *The process $(\mathcal{Z}_k)$ converges in distribution under the law of μ_φ to $\mathcal{Z} = h(\mathcal{B})$.*

Proof. We have $\mathcal{Z}_k = h(\mathcal{Y}_k)$, and by continuity we get that $\mathcal{Z}_k$ converges in distribution to $h(\mathcal{B})$. □

Since $\mathcal{Z}_k$ converges to $\mathcal{Z}$ in distribution and Φ_k converges to (the deterministic) Φ in probability, the couple $(\mathcal{Z}_k, \Phi_k)$ converges to $(\mathcal{Z}, \Phi)$ ([Bil99], Theorem 3.9). By continuity of the composition we conclude that $\mathcal{Z}_k \circ \Phi_k$ converges in distribution to $\mathcal{Z} \circ \Phi$. By continuity again we finally get that $\mathcal{X}_k$ converges in distribution to

$$\mathcal{X} = \gamma(h(\mathcal{B}) \circ \Phi).$$

3.2.2 The limit is a Wiener process.

To finish the proof we are left to characterize the limiting process $\mathcal{X}$. Denote the transfer matrix by $U = (u_{ij})$. Note that $\theta_1 < \theta_2$. For any $t \in [0,1]$ we have

$$\begin{aligned}\mathcal{X}(t) &= \gamma(h(\mathcal{B}) \circ \Phi)(t)\\ &= h_1(U\mathcal{W})(\theta_1 t, \theta_2 t) + h_2(U\mathcal{W})(\theta_1 t, \theta_2 t)\\ &= u_{11}\sigma_1 W_1(\theta_1 t) + u_{12}\sigma_2 W_2(\theta_1 t) + u_{21}\sigma_1 W_1(\theta_2 t) + u_{22}\sigma_2 W_2(\theta_2 t)\\ &= (u_{11} + u_{21})\sigma_1 W_1(\theta_1 t) + (u_{12} + u_{22})\sigma_2 W_2(\theta_1 t) +\\ &\qquad + u_{21}\sigma_1(W_1(\theta_2 t) - W_1(\theta_1 t)) + u_{22}\sigma_2(W_2(\theta_2 t) - W_2(\theta_1 t))\end{aligned}$$

By independence of the processes W_i and independence of their increments, we get that $\mathcal{X}(t)$ is again a Wiener process, its variance is

$$\begin{aligned}\sigma^2 &:= \operatorname{var} \mathcal{X}(1)\\ &= ((u_{11} + u_{21})\sigma_1)^2\theta_1 + ((u_{12} + u_{22})\sigma_2)^2\theta_1 + (u_{21}\sigma_1)^2(\theta_2 - \theta_1) + (u_{22}\sigma_2)^2(\theta_2 - \theta_1).\end{aligned} \tag{8}$$

Remark 3. We remark that the variance vanishes if and only if

$$\begin{cases} u_{11}\sigma_1 + u_{21}\sigma_1 &= 0\\ u_{12}\sigma_2 + u_{22}\sigma_2 &= 0\\ u_{21}\sigma_1 &= 0\\ u_{22}\sigma_2 &= 0\end{cases} \iff U\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = 0,$$

which is equivalent to $\sigma_1 = \sigma_2 = 0$ since the matrix U is invertible. This is equivalent to the fact that the covariance matrix $Q = 0$, which happens if and only if both ϕ_1 and ϕ_2 are cohomologous to zero. Then, we use Proposition 2.16.

We finally have the conclusion: the process N_ε converges in the Skorohod topology to a Wiener process N with variance σ^2.

4 Generalizations and open questions

For each of these situations the method developed in the paper gives a version of the theorem. We do not rewrite their proofs in full details since it is very close.

4.1 Higher dimension case

In this subsection we briefly indicate how we have to adapt the proof to the higher dimension case: we consider the d dimensional torus $\mathbb{T}^d = \left(\mathbb{R}/\mathbb{Z}\right)^d$. We denote by π_k the canonical projections $\pi_k(x_1, \ldots, x_d) = (x_1, \ldots, x_k)$.

Definition 4.1. *A map $T : \mathbb{T}^2 \circlearrowleft$ is said to be a* skew product *if it is of the form $T(x) = (f_1(x_1), f_2(x_1, x_2), \ldots, f_d(x_1, \ldots, x_d))$.*

Theorem 4.1. *Let $T : \mathbb{T}^d \circlearrowleft$ be a skew product $\mathcal{C}^2$ expanding map. Let φ be a Hölder continuous function from $\mathbb{T}^d$ to $\mathbb{R}$. Let μ_φ be the equilibrium state associated to φ. Let $\delta := \delta_{\mu_\varphi}$ be its Hausdorff dimension.*

We assume that the sequence

$$\lambda_{\mu_\varphi,i} := \int \log \left| \frac{\partial f_i}{\partial x_i} \right| \circ \pi_i d\mu_\varphi, \quad i = 1, \ldots, d$$

is increasing. Then there exists a real number $\sigma \geq 0$ such that the process

$$\frac{\log \mu_\varphi \left(B(x, \varepsilon^t) \right) - t\delta \log \varepsilon}{\sqrt{-\log \varepsilon}}, \quad t \in [0, 1]$$

converges in $\mathcal{D}([0, 1])$ in distribution under the law of μ_φ to the process $\sigma W(t)$, where W is the standard Wiener process.

In addition, the variance σ^2 is zero if and only if μ_φ is the unique absolutely continuous invariant measure, or equivalently φ is cohomologous to $-\log |\det DT|$.

As we said above, the hypothesis $\lambda_{\mu,i} < \lambda_{\mu,i+1}$ means that we are dealing with the non-conformal case.

The proof is done by induction. The expression $T(x, y) = (f(x), g(x, y))$ is still valid if we consider y as a point in $\mathbb{T}$ and x as a point in $\mathbb{T}^{d-1}$.

The main steps to adapt are the following:

Lemma 2.1 to construct a family of partitions, defined by sectors in $\mathbb{T}^d$.

Lemma 2.2 to prove that the Lyapunov exponents are exactly the quantities $\lambda_{\mu_\varphi,i}$. Here the one-dimension argument does not hold but the result follows from the fact that the differential has triangular matrix with values $\log \left| \frac{\partial f_i}{\partial x_i} \right| \circ \pi_i$ on the diagonal. The fact that the μ_φ-integrals of these quantities are ordered yields that the Lyapunov exponents are the $\lambda_{\mu_\varphi,i}$.

Lemma 2.3 to control the slope of the hyperplane $T^{-n}(1, \ldots, 1, 0)$, and by induction, the angle of the Lyapunov direction E_{i+1} with respect to the partial Oseledec splitting $E_1 \oplus \ldots \oplus E_i$.

Definition 2.4 to define multi-temporal approximation of maps. Note that $n_\varepsilon(x, y)$ is well-defined. The quantity $m_\varepsilon(x)$ is then replaces by quantities of the form $n_\varepsilon(x_1, \ldots, x_{d-1}), n_\varepsilon(x_1, \ldots, x_{d-2}), \ldots, n_\varepsilon(x_1, x_2), m_\varepsilon(x_1)$.

Lemma 2.6: for each k, $\lim_{\varepsilon \to 0} \frac{n_\varepsilon(x_1,\ldots,x_k)}{-\log \varepsilon} = \frac{1}{\lambda_{\mu_\varphi,d-k+1}}$. An important fact is that these limit are deterministic.

Proposition 2.12 is still valid. Note that we only need to project once: $\pi_{k-d-1} = \pi_{k,k-1} \circ \pi_k$, where $\pi_{i,i-1}$ is the canonical projection from $\mathbb{T}^i$ onto $\mathbb{T}^{i-1} \approx \mathbb{T}^{i-1} \times \{0\}$.

The WIP principle (Theorem 3.1) holds in higher dimension case.

Then we use the random change of time (Lemma 3.1).

4.2 Conformal hyperbolic dynamics

We present two situations of conformal hyperbolic dynamics where our method can be applied verbatim. We refer to [Bar08] for their precise definitions, and also for the estimates concerning the geometry of cylinders and further notions such as invariant measures of full dimension and maximal dimension.

Theorem 4.2. *Let J be a repeller of a $C^{1+\alpha}$ transformation T, for some $\alpha > 0$, such that T is conformal and topologically mixing on J, and μ be the equilibrium measure of a Hölder continuous $\varphi\colon J \to \mathbb{R}$. Denote the asymptotic variance of $\varphi + \frac{h_{\mu_\varphi}}{\lambda_{\mu_\varphi}} \log f'$ by σ_u^2.*

Then the statement of the main theorem holds. The variance of the limit is $\sigma^2 := \frac{\sigma_u^2}{\lambda_{\mu_\varphi}}$, which vanishes iff μ is the measure of maximal (or full) dimension in J.

The result is obtained by a simplification of our proof: just remove any dependence in y. In particular, one can use formula (8) with $u_{21} = u_{22} = u_{12} = 0$.

Theorem 4.3. *Let Λ be a locally maximal hyperbolic set of a $C^{1+\alpha}$ diffeomorphism T, for some $\alpha > 0$, such that T is conformal and topologically mixing on Λ, and μ be the equilibrium measure of a Hölder continuous $\varphi\colon \Lambda \to \mathbb{R}$.*

Denote the asymptotic variance of $\varphi + \frac{h_{\mu_\varphi}}{\lambda_s} \log \|df|E^s\|$ by σ_s^2. Denote the asymptotic variance of $\varphi + \frac{h_{\mu_\varphi}}{\lambda_u} \log \|df|E^u\|$ by σ_u^2.

Then the statement of the main theorem holds. The variance of the limit is $\sigma^2 := \frac{\sigma_s^2}{\lambda_s} + \frac{\sigma_u^2}{\lambda_u}$, which vanishes iff μ has full dimension in Λ.

Remark 4. Although there always exists an invariant measure of maximal dimension in Λ, it is unlikely that Λ supports an invariant measure with full dimension. Indeed, we generically have that $\sup_\mu \dim_H \mu < \dim_H(\Lambda)$.

An interesting situation is for the SRB, or physical measure. When Λ is the whole manifold then generically the SRB measure does not have full dimension, in particular the variance $\sigma^2 \neq 0$.

The proof here is somehow different. The key point is that there are local product structures, both for coordinates (see *e.g.* [Bow75]) and for Gibbs measures (see *e.g.* [Lep00]). Moreover, if we locally set

$$\mu_\varphi \approx \mu_\varphi^s \otimes \mu_\varphi^u,$$

these two measures μ_φ^u and μ_φ^s also satisfy some Gibbs property.

Using these local coordinates, a ball $B((x,y),\varepsilon)$ can be approximate by a cylinder of the form

$$C_{-m_\varepsilon(x)}^{n_\varepsilon(y)}.$$

It is important here to note that the quantity n_ε depends only on the future (the unstable direction, coordinate y) and conversely, $-m_\varepsilon$ depends only on the past (the stable direction, coordinate x). Then, using the local product structure for the Gibbs measures we get

$$\mu_\varphi(B(x,y),\varepsilon) \approx S_{n_\varepsilon(y)}(\phi_u)(y) + S_{m_\varepsilon(x)}(\phi_s)(x), \tag{9}$$

with ϕ_s ad ϕ_u Hölder continuous, both cohomologous to φ, and depending only on past (resp. future) coordinates. We also observe that the asymptotic distributions of both terms are independent. Then adapt Section 3.

4.3 Possible extensions to other dynamical systems

Our main hypotheses was the uniform expansion and skew product structure. It seems however that these hypotheses can be relaxed and we discuss this point below.

4.3.1 Non-uniformly expanding maps of an interval.

The first possibility is to relax the uniformity in the expansion. There is a vast and still growing literature in this subject. However, these results mainly concern absolutely continuous invariant measures. As already said, these measures have no fluctuations and our result is irrelevant in these cases. For other potentials, the literature is not so large. Basically our method could be applied in principle for maps and their Gibbs measures, such that the (functional) CLT hold for a sufficiently regular class of observables.

It is not clear for the moment if the method could be adapted to conformal "mostly expanding maps" as studied by Oliveira and Viana in [OV08]. Note that for these maps, the equilibrium state is not a Gibbs measure but only a non-lacunar Gibbs measure. This seems to be an obstruction to adapt our method.

We emphasize that for some non-uniformly expanding maps the CLT does not hold in the classical form; for example $S_n\phi$ could be in the non-standard basin of attraction of the normal law; in that case we could prove a version of our main theorem with a suitable modification of the normalization. A more difficult task is when we have a convergence to a stable law of some index $\alpha < 2$. In that case we believe that our method could be carried out, but some difficulties may arise due to the discontinuity of the paths in non Brownian Levy process.

4.3.2 Non-conformal without skew product structure

The second and most challenging situation is for non-conformal maps without the skew product structure. Note that we used two strong consequences of this structure: 1) the Lyapunov splitting exists, without going through a natural extension and 2) the projected measure has the Gibbs property. Still, we believe that the result remains true in general.

Conjecture. *Let M be a compact smooth Riemannian manifold and $T : M \circlearrowleft$ be an Axiom-A diffeomorphism. Let φ be a Hölder continuous function from M to $\mathbb{R}$. Let μ_φ be the equilibrium state associated to φ. Let δ be its Hausdorff dimension.*

Then there exists a real number $\sigma \geq 0$ such that the process

$$\frac{\log \mu_\varphi\left(B(x,\varepsilon^t)\right) - t\delta\log\varepsilon}{\sqrt{-\log\varepsilon}}$$

converges in $\mathcal{D}([0,1])$ and in distribution to the process $\sigma W(t)$, where W is the standard Wiener process.

In particular we believe that the SRB measure of a topologically mixing Anosov diffeomorphism of a compact Riemaniann manifold should enjoy this property, and that the variance will vanishes iff the measure is absolutely continuous.

References

[Bar08] Luis Barreira. *Dimension and recurrence in hyperbolic dynamics*, volume 272 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2008.

[BF09] J. Barral and D.J. Feng. Weighted thermodynamic formalism and applications. *Preprint*, 2009.

[Bil99] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[Bow75] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, volume 470 of *Lecture notes in Math.* Springer-Verlag, 1975.

[BPS99] L. Barreira, Y. Pesin, and J. Schmeling. Dimension and product structure of hyperbolic measures. *Ann. of Math. (2)*, 149(3):755–783, 1999.

[CU09] Jean-Rene Chazottes and Edgardo Ugalde. On the preservation of gibbsianness under symbol amalgamation. to appear in a special volume published by Cambridge Univ. Press, 2009.

[GP97] Dimitrios Gatzouras and Yuval Peres. Invariant measures of full dimension for some expanding maps. *Ergodic Theory Dynam. Systems*, 17(1):147–167, 1997.

[KP09] T. Kempton and M. Pollicott. Factors of gibbs measures for full shifts. to appear in the proceedings of the BIRS Workshop on Hidden Markov Processes (CUP), 2009.

[Led81] F. Ledrappier. Some relations between dimension and Lyapounov exponents. *Comm. Math. Phys.*, 81(2):229–238, 1981.

[Lep00] R. Leplaideur. Local product structure for equilibrium states. *Trans. Amer. Math. Soc.*, 352(4):1889–1912, 2000.

[Luz06] Nuno Luzia. A variational principle for the dimension for a class of non-conformal repellers. *Ergodic Theory Dynam. Systems*, 26(3):821–845, 2006.

[LY85a] F. Ledrappier and L.-S. Young. The metric entropy of diffeomorphisms Part I: Characterization of measures satisfying Pesin's entropy formula. *Annals of Mathematics*, 122:509–539, 1985.

[LY85b] F. Ledrappier and L.-S. Young. The metric entropy of diffeomorphisms Part II: Relations between entropy,exponents and dimension. *Annals of Mathematics*, 122:540–574, 1985.

[McM84] Curt McMullen. The Hausdorff dimension of general Sierpiński carpets. *Nagoya Math. J.*, 96:1–9, 1984.

[MN09] Ian Melbourne and Matthew Nicol. A vector-valued almost sure invariance principle for hyperbolic dynamical systems. *Ann. Probab.*, 37(2):478–505, 2009.

[OV08] Krerley Oliveira and Marcelo Viana. Thermodynamical formalism for robust classes of potentials and non-uniformly hyperbolic maps. *Ergodic Theory Dynam. Systems*, 28(2):501–533, 2008.

[Rue04] David Ruelle. *Thermodynamic formalism*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2004. The mathematical structures of equilibrium statistical mechanics.

[Sau06] Benoît Saussol. Recurrence rate in rapidly mixing dynamical systems. *Discrete Contin. Dyn. Syst.*, 15(1):259–267, 2006.