RECURRENCE SPECTRUM IN SMOOTH DYNAMICAL SYSTEMS.

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Abstract. We prove that for conformal expanding maps the return time does have constant multifractal spectrum. This is the counterpart of the result by Feng and Wu in the symbolic setting.

1. Introduction

Let $f : \mathcal{M} \to \mathcal{M}$ be a map on a smooth compact manifold $\mathcal{M}$ with metric d. Let $\Lambda \subset \mathcal{M}$ be a compact f-invariant set. For $x \in \mathcal{M}$ we can define the return times in r-neighborhoods by

$$\tau_r(x) = \inf\{n > 0\colon d(f^n x, x) < r\}.$$

We put $\tau_r(x) = \infty$ if the orbit of x does not come back close to x.

When $\mathcal{M}$ is an interval, f is a piecewise monotonic transformation with some regularity (piecewise $C^{1+\alpha}$ suffices) and μ is an f-invariant ergodic measure with nonzero entropy, Saussol, Troubetzkoy and Vaienti [11], building on results by Ornstein and Weiss [10] and Hofbauer and Raith [5, 6], proved that

$$\lim_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = \dim_H \mu \quad \text{for } \mu\text{-almost every } x \tag{1}$$

where $\dim_H \mu$ stands for the Hausdorff dimension of the measure μ. Barreira and Saussol established, with completely different techniques, Equation (1) for equilibrium states preserved by repellers [2] and Axiom A diffeomorphisms [1] in any dimension.

From now on we assume that $\Lambda \subset \mathcal{M}$ is a *conformal repeller* of a $C^{1+\alpha}$ map $f\colon \mathcal{M} \to \mathcal{M}$. That is to say that Λ is a compact forward invariant set and that f is $C^{1+\alpha}$ on $\mathcal{M}$, expanding and conformal on Λ. Moreover we suppose that there exists an open set $V \subset \mathcal{M}$ such that $\Lambda = \bigcap_{n\ge 0} f^{-n}V$.

From the multifractal analysis of conformal expanding maps it is well known that for any real $s \in [0, \dim_H \Lambda]$ there exists an ergodic equilibrium state μ_s (in fact there are many such measures for non extremal values of s). Therefore Equation (1) together with basic properties of dimension of measures imply that

$$\dim_H \left\{ x \in \Lambda\colon \lim_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \text{ exists and equals } s \right\} \ge \dim_H \mu_s = s. \tag{2}$$

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The reader familiar with multifractal analysis will believe that in fact the dimension of this set is larger than this, and at this point may expect a nice non-trivial (strictly) concave analytic curve. In fact, this is really not the case, as our first theorem says

Theorem 1. *Let $\Lambda \subset \mathcal{M}$ be a conformal repeller of the $C^{1+\alpha}$ map $f\colon \mathcal{M} \to \mathcal{M}$. For any $\alpha \le \beta$ in $[0,\infty]$ we have*

$$\dim_H \left\{ x \in \Lambda\colon \varliminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = \alpha \text{ and } \varlimsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = \beta \right\} = \dim_H \Lambda.$$

Theorem 1 implies in particular that for all values of $s \in [0,\infty]$ the dimension in the left hand side of Equation (2) equals $\dim_H \Lambda$.

The corresponding result of Theorem 1 in the symbolic case was established by Feng and Wu [3]. Although our method relies on some essential ideas present in the above mentioned paper, we emphasize that our proof is not a direct translation of the symbolic result. [1]

In Section 2 we prove the theorem in the situation where the system (f,Λ) is conjugated to a subshift of finite type. It happens that our problem can be reduced to this case and this is proven in Section 3.

Contents

2. System conjugated to a subshift of finite type

In this section we give a proof of Theorem 1 in a special case.

Theorem 2. *Let $\Lambda \subset \mathcal{M}$ be such that (f,Λ) is conjugated to a subshift of finite type. Then the result of Theorem 1 holds.*

The proof of this result is decomposed in several parts. First we give a construction similar to the one by Feng and Wu of points with prescribed recurrence in a symbolic space [3].

[1] After the completion of this work we learn that L. Olsen has obtained — independently and with different techniques — a similar result for conformal Iterated Function Systems satisfying some strong separation condition, which is equivalent to a conjugacy to a subshift of finite type, the case considered in Theorem 2.

2.1. **Producing points with given symbolic recurrence sequence.** For a given sequence of integer (ℓ_n) such that

(a) $\exists n_0, \ell_{n+1} \geq \ell_n + 2n$ for any $n \geq n_0$ and

(b) $\lim_{n\to\infty} \ell_n/n^2 = \infty$,

we define a function g nearly as in [3] such that points in the image of g have (ℓ_n) as recurrence sequence. Let $\mathcal{A} \subset \mathcal{B}$ be two alphabets (i.e. finite or countable sets) with $\mathcal{A} \neq \mathcal{B}$ and $\sharp\mathcal{B} \geq 3$. Choose a *marker* $m \in \mathcal{B} \setminus \mathcal{A}$.

Definition 3. Let $c \neq \bar{c} \in \mathcal{B}\setminus\{m\}$. Define $g\colon \mathcal{A}^{\mathbb{N}} \to \mathcal{B}^{\mathbb{N}}$ by $g(w) = \lim w^{(k)}$, where the sequence $w^{(k)}$ is constructed recursively as follows.

We put $w^{(0)} = w^{(1)} = \cdots = w^{(n_0-1)} = mw_1w_2\cdots$. For $k \geq n_0$ we construct $w^{(k)}$ from $w^{(k-1)}$ by inserting the block $w_1^{(k-1)}\cdots w_k^{(k-1)}y_k$ at the position $\ell_k + 1$, i.e.

$$w^{(k)} = w_1^{(k-1)}w_2^{(k-1)}\cdots w_{\ell_k}^{(k-1)}w_1^{(k-1)}\cdots w_k^{(k-1)}y_k w_{\ell_k+1}^{(k-1)}w_{\ell_k+2}^{(k-1)}\cdots ,$$

where $y_k = c$ if $w_{k+1}^{(k-1)} \neq c$ or $y_k = \bar{c}$ if $w_{k+1}^{(k-1)} = c$, so that $y_k \neq m, w_{k+1}^{(k-1)}$. Note that all the $w^{(k)}$ share the same ℓ_{n+1} first letters whenever $k \geq n$, hence the limit $g(w)$ is well defined. □

Let us define the k-repetition time of w by

$$R_k(w) = \inf\{n > 0\colon w_{n+1}w_{n+2}\cdots w_{n+k} = w_1w_2\cdots w_k\}.$$

The main interest of the function g resides in the following

Lemma 4 ([3]). *If $w \in \mathcal{A}^{\mathbb{N}}$ and $k \geq n_0$ then $R_k(g(w)) = \ell_k$.*

Proof of Lemma 4. We do it by induction. Let $w^* = g(w)$. Since there is no m in w it is obvious that $R_{n_0}(w^*) = \ell_{n_0}$. Assume that for some $k \geq n_0$ we have proven that $R_k(w^*) = \ell_k$. By construction this implies that $\ell_k < R_{k+1}(w^*) \leq \ell_k + k$ or $R_{k+1}(w^*) = \ell_{k+1}$ (note that there can be no m between the positions $\ell_k + k + 1$ and ℓ_{k+1} in w^*). Suppose that we are in the first case and let $p = R_{k+1}(w^*)$. The number of m in the block $w_{p+1}^*\cdots w_{p+k}^* = w_1^*\cdots w_k^*$ being equal to the number of m inside the block $w_{\ell_k+1}^*\cdots w_{\ell_k+k}^*$ we get a contradiction if $p > \ell_k$ (because $w_{\ell_k+1}^* = m$ is missed and there is no m in $w_{\ell_k+k+1}^*\cdots w_{p+k}^*$). □

Although one can imagine similar constructions for subshifts of finite type, we decided to work with fullshifts to make the core argument more transparent. However, we then have to work a bit to apply it to our systems, which are presumably only coded by subshifts of finite type. This is the aim of next section.

2.2. **Building the source of large dimension.** Suppose that $\dim_H \Lambda > 0$ otherwise there is nothing to prove in Theorem 2. Let (Σ, T) be a subshift of finite type conjugated to (f, Λ) and denote by $\pi\colon \Sigma \to \Lambda$ the conjugating homeomorphism. For our purpose there is no loss of generality to suppose that (Σ, T) is topologically mixing. Thus at least one letter, say a, has two successors b and c. Let then B and C be the shortest paths (and smallest in the lexicographic order) starting in b and c, respectively, and ending in a. That is aB and aC are the smallest words of Σ with prefix ab and ac, respectively, and suffix a. It is clear that there is a one-to-one correspondence between the set of words and the set of cylinders. Without confusion, we also use a word to denote it's corresponding cylinder. Let $A = aB$ and denote by $t(\omega)$ the return time of $\omega \in A$ into A,

$$t(\omega) = \inf\{t > 0\colon T^t\omega \in A\}.$$

Note that t is unbounded on A since there is a disjoint union contained in A. More precisely,

$$A \supset aBB \cup aBCB \cup aBCCB \cup \cdots .$$

To simplify the exposition we will consider the new alphabet consisting in $|A|$-cylinders, where $|A|$ denotes the length of the cylinder A, together with its associated transition matrix. Let then $\mathcal{Z}_n$ be the partition of Σ by n-cylinders (the new ones), and denote by $\mathcal{F}_n$ the finite σ-algebra generated by $\mathcal{Z}_n$.

Let μ be the equilibrium state of the potential $\varphi = -\dim_H(\Lambda)\log|Df|\circ\pi$. It is well known that μ is the measure of maximal dimension and the Bowen pressure formula gives $P(\varphi|\Sigma) = 0$. In addition the measure μ is ψ-mixing (See e.g. [7] for properties of equilibrium states), which means that

$$\psi(n) \stackrel{\text{def}}{=} \sup_m \sup_{U\in\mathcal{F}_m, V\in\mathcal{B}} \left| \frac{\mu(U\cap T^{-n-m}V)}{\mu(U)\mu(V)} - 1 \right| \underset{n\to\infty}{\longrightarrow} 0.$$

Hence there exists m_0 such that

$$\mu(A\cap T^{-m_0}(\Sigma\setminus A)\cap T^{-2m_0}(\Sigma\setminus A)\cap\cdots\cap T^{-mm_0}(\Sigma\setminus A)) \le \delta^m,$$

where $\delta \stackrel{\text{def}}{=} \mu(\Sigma\setminus A)(1+\psi(m_0)) < 1$. Set $H_n = \{\omega\in A\colon t(\omega)\ge n\}$. The preceding estimate yields

$$\mu(H_n) \le \delta^{n/m_0}.$$

Moreover we have $H_n = A\setminus\{x\in A\colon t(x)\le n-1\}\in\mathcal{F}_n$. From Proposition 7 (see Section 3 below), setting $\Sigma_n = (\Sigma\setminus H_n)^\infty$, we get that

$$\lim_{n\to\infty} P(\varphi|\Sigma_n) = P(\varphi|\Sigma) = 0. \tag{3}$$

Let ν_n be an ergodic equilibrium state of φ on Σ_n. If n is sufficiently large then $\nu_n(A) > 0$ (otherwise ν_n would be supported on the forward invariant set $(\Sigma\setminus A)^\infty$, on which the pressure of φ is strictly less than $P(\varphi|\Sigma)$).

Let $\hat\Sigma\subset A$ denotes the set of points returning infinitely many times in A. Let us define the induced system $(\hat\Sigma,\hat T,\hat\nu_n)$ by

$$\hat T(x) = T^{t(x)}(x), \quad \hat\nu_n = \frac{1}{\nu_n(A)}\nu_n|_A.$$

Since $\hat T$ is the induced transformation on a cylinder of a subshift of finite type, $(\hat\Sigma,\hat T)$ is a full shift with the countable alphabet

$$\hat{\mathcal{Z}} = \bigcup_{m=1}^\infty \{Z\cap\hat\Sigma\colon Z\in\mathcal{Z}_{m+1}, Z\subset A, t|_Z = m\}.$$

Since the return times in A are bounded (by $n-1$) on $\Sigma\setminus H_n$, one has

$$\hat\Sigma_n \stackrel{\text{def}}{=} A\cap\Sigma_n \subset \hat\Sigma.$$

Observe that $(\hat\Sigma_n, \hat T)$ is still a fullshift over the finite alphabet

$$\hat{\mathcal{Z}}^n = \bigcup_{m=1}^{n-1} \{Z \cap \hat\Sigma_n \colon Z \in \mathcal{Z}_{m+1}, Z \subset A, t|_Z = m\}.$$

Set $\psi = \log|Df| \circ \pi$, $S_k\psi(\omega) = \sum_{j=0}^{k-1} \psi(T^j\omega)$ and $\hat S_k t(\omega) = \sum_{j=0}^{n-1} t(\hat T^j \omega)$. Let us define *the source of large dimension* by

$$G_n = \left\{ \omega \in \hat\Sigma_n \colon (i) \lim_{k\to\infty} \frac{1}{k} S_k\psi(\omega) \to \nu_n(\psi),\ (ii) \lim_{k\to\infty} \frac{1}{k} \hat S_k t(\omega) \to \hat\nu_n(t) \right\}.$$

By Birkhoff's Ergodic Theorem we have $\hat\nu_n(G_n) = 1$ which provides the following lower bound

$$\dim_H \pi G_n \geq \dim_H \pi\hat\nu_n = \dim_H \pi\nu_n|_A = \dim_H \pi\nu_n. \tag{4}$$

The last equality follows by ergodicity of the equilibrium state ν_n. Moreover,

$$\dim_H \pi\nu_n = \frac{h_{\nu_n}}{\int \psi d\nu_n} = \dim_H \Lambda + \frac{P(\varphi|\Sigma_n)}{\int \psi d\nu_n}.$$

Observe that this together with (3) and (4) imply that

$$\lim_{n\to\infty} \dim_H \pi G_n = \dim_H \Lambda. \tag{5}$$

2.3. **The image of the source has good symbolic recurrence.** With the alphabets $\mathcal{A} = \hat{\mathcal{Z}}^n$ and $\mathcal{B} = \hat{\mathcal{Z}}$ we can use Definition 3 to produce a function $g \colon \hat\Sigma_n \to \hat\Sigma$. For the marker $m \in \mathcal{B} \setminus \mathcal{A}$ we choose a $m \in \hat{\mathcal{Z}}$ such that $t|_m = n$.

Given an integer k, let p be such that $\ell_p \leq k < \ell_{p+1}$ and set $\epsilon_k = (p+2)^2/\ell_p$. The property (b) of the sequence (ℓ_p) implies that $\epsilon_k \to 0$ as $k \to \infty$.

Observe that t is $\hat{\mathcal{Z}}$ measurable, hence for any $\omega \in G_n$ we have

$$|\hat S_k t(g\omega) - \hat S_k t(\omega)| \leq k\epsilon_k n, \tag{6}$$

since the symbolic sequence of $g(\omega)$ and ω differs only because of the block of length i inserted at position ℓ_i, which makes a difference of at most $\sum_{i=n_0+1}^{p+1} i \leq k\epsilon_k$ letters. Hence for any $\omega \in G_n$ we have

$$\lim_{k\to\infty} \frac{1}{k} \hat S_k t(g\omega) = \hat\nu_n(t). \tag{7}$$

Notice that a k-cylinder of $(\hat\Sigma, \hat T)$ containing ω is indeed a $\hat S_k t(\omega)$ cylinder of (Σ, T). Hence Lemma 4 gives that for $\omega \in G_n$ we have (the notation $\hat R_k$ stands for the k-repetition time computed with $\hat T$)

$$R_{\hat S_k t(g\omega)}(g\omega) = \hat R_k(g\omega) = \ell_k.$$

Passing to the limit using Equation (7) and the fact that the sequence $(R_h)_h$ is monotone gives us that the image gG_n has good symbolic recurrence. Namely, for any $\omega \in G_n$,

$$\underline{\lim}_{h\to\infty} \left(\overline{\lim}_{h\to\infty} \right) \frac{\log R_h(g\omega)}{h} = \underline{\lim}_{k\to\infty} \left(\overline{\lim}_{k\to\infty} \right) \frac{1}{\hat\nu_n(t)} \frac{\log \ell_k}{k}. \tag{8}$$

2.4. **The image has large dimension.** The fact that the dimension of the image $\pi g G_n$ is large will follow from (5) if we are able to prove that the inverse function of $\gamma \stackrel{\text{def}}{=} \pi\circ g\circ\pi^{-1}$ has some smoothness.

Let $\omega\in G_n$. Denotes the h-cylinder of $(\Sigma,T,\mathcal{Z})$ containing $g(\omega)$ by $\mathcal{Z}(g\omega,h)$. If k is such that

$$\hat S_k t(g\omega)\le h<\hat S_{k+1}t(g\omega)$$

then

$$\mathcal{Z}(g\omega,h)\subset\hat{\mathcal{Z}}(g\omega,k),$$

where $\hat{\mathcal{Z}}(z,k)$ denotes the k-cylinder of $(\hat\Sigma,\hat T,\hat{\mathcal{Z}})$ containing a point z. By construction of g we have

$$g^{-1}\hat{\mathcal{Z}}(g\omega,k)\subset\hat{\mathcal{Z}}(\omega,k(1-\epsilon_k)).$$

Because of (6) we have

$$\hat{\mathcal{Z}}(\omega,k(1-\epsilon_k))\subset\mathcal{Z}(\omega,h(1-\delta_h))$$

for some sequence $\delta_h\to0$ as $h\to\infty$ (independent of ω). Putting together this chain of inclusions gives us that

$$g^{-1}\mathcal{Z}(g\omega,h)\subset\mathcal{Z}(\omega,h(1-\delta_h)). \tag{9}$$

This estimate on cylinders may be exploited for balls using the conjugacy assumption.

Proposition 5. *Let (f,Λ) be a $C^{1+\alpha}$ conformal expanding map conjugated to a subshift. Then there exists a constant $\kappa\in(0,1)$ such that for any integer n and any $x\in\Lambda$*

$$B(x,\kappa|D_xf^n|^{-1})\cap\Lambda\subset\pi\mathcal{Z}(\pi^{-1}x,n)\subset B(x,\kappa^{-1}|D_xf^n|^{-1})\cap\Lambda. \tag{10}$$

Proof. We give only a sketch. Assume that $\operatorname{diam}\Lambda=1$ and $|D_xf|>1$ for any $x\in\Lambda$. Since f is conjugated to a subshift the minimal distance between two image πZ and $\pi Z'$ of different elements $Z,Z'\in\mathcal{Z}$ is bounded from below by some constant $\delta>0$. Moreover, by distortion there exists a constant D such that for any n and x,y in the same n-cylinder we have $|D_xf^n|\le D|D_yf^n|$. Let $\kappa=\delta/D$.

Suppose that $d(x,y)<\kappa|D_xf^n|^{-1}$. This implies that $d(x,y)<\delta$, thus $\mathcal{Z}(\pi^{-1}x,1)=\mathcal{Z}(\pi^{-1}y,1)$. If $\mathcal{Z}(\pi^{-1}x,k-1)=\mathcal{Z}(\pi^{-1}y,k-1)$ for some $k<n$, then we get $d(f^kx,f^ky)\le D|D_xf^k|d(x,y)<\delta$. Hence $\mathcal{Z}(\pi^{-1}x,k)=\mathcal{Z}(\pi^{-1}y,k)$. By induction we get the first inclusion in Equation (10). The remaining inclusion is easier and we omit its proof. □

Lemma 6. *The function γ^{-1} is α-Hölder continuous on $g(G_n)$ for any $\alpha\in(0,1)$.*

Proof. By arguments similar to those in the proof of Equation (6), but using the continuity of ψ, we get that

$$\lim_{n\to\infty}\sup_{\omega\in G_n}\left|\frac1n S_n\psi(\omega)-\frac1n S_n\psi(g\omega)\right|=0. \tag{11}$$

Let $x,y\in\pi G_n$. Let n be such that

$$\kappa\exp(-S_{n+1}\psi(g\pi^{-1}x))\le d(\gamma x,\gamma y)<\kappa\exp(-S_n\psi(g\pi^{-1}x)).$$

It follows from Proposition 5 that $g\pi^{-1}y \in \mathcal{Z}(g\pi^{-1}x, n)$. Thus $\pi^{-1}y \in \mathcal{Z}(\pi^{-1}x, n(1-\delta_n))$ by (9). Using Proposition 5 again we find that

$$d(x,y) < \kappa^{-1}\exp(-S_{n(1-\delta_n)}\psi(\pi^{-1}x)).$$

For any $\alpha \in (0,1)$, because of (11), the facts that $\delta_n \to 0$, ψ is strictly positive and bounded, we can find an $n(\alpha)$ such that for all $n \geq n(\alpha)$ and all x we have

$$\exp(-S_{n(1-\delta_n)}\psi(\pi^{-1}x)) \leq \kappa^2 \exp(-\alpha S_{n+1}\psi(g\pi^{-1}x))$$

This shows that whenever $d(\gamma x, \gamma y) < \kappa \inf \exp S_{n(\alpha)}\psi$ we have

$$d(x,y) < d(\gamma x, \gamma y)^{\alpha},$$

proving the lemma. □

Since $\pi G_n = \gamma(\pi g G_n)$ Lemma 6 ensures that the image of the source has large dimension:

$$\dim_H \pi g(G_n) \geq \dim_H \pi G_n. \tag{12}$$

2.5. **The image has good recurrence rates.** By Proposition 5 we have

$$\tau_{\kappa\exp(S_k\psi(\pi^{-1}x))}(x) \geq R_k(\pi^{-1}x) \geq \tau_{\kappa^{-1}\exp(S_k\psi(\pi^{-1}x))}(x), \tag{13}$$

for any $x \in \Lambda$. Let $\lambda_n = \int \psi d\nu_n$. By Equation (11), g does not change the Birkhoff average of ψ thus for any $x \in \pi g(G_n)$, by the property (ii) in the definition of G_n we have

$$\lim_{k\to\infty} \frac{1}{k} S_k\psi(\pi^{-1}x) = \lambda. \tag{14}$$

Then we choose the sequence (ℓ_k) as in [3] such that

$$\underline{\lim}_{k\to\infty} \frac{\log \ell_k}{k} = \alpha\lambda_n\hat{\nu}_n(t) \quad \text{and} \quad \overline{\lim}_{k\to\infty} \frac{\log \ell_k}{k} = \beta\lambda_n\hat{\nu}_n(t).$$

Using the bounds in (13) and (14) we get that for any $x \in \pi g(G_n)$

$$\underline{\lim}_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = \underline{\lim}_{k\to\infty} \frac{\log R_k(\pi^{-1}x)}{\lambda_n k} = \underline{\lim}_{k\to\infty} \frac{1}{\lambda_n\hat{\nu}_n(t)} \frac{\log \ell_k}{k} = \alpha,$$

where the second equality follows from (8). The same arguments for the $\overline{\lim}$ gives that $x \in E(\alpha,\beta)$, where

$$E(\alpha,\beta) = \left\{ x \in \Lambda\colon \underline{\lim}_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = \alpha \text{ and } \overline{\lim}_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = \beta \right\}. \tag{15}$$

Since $\pi g(G_n) \subset E(\alpha,\beta)$, Theorem 2 follows from (5) and (12).

3. Reduction to a system conjugated to a subshift

We want to ensure that the original system contains subsystems with arbitrary large dimension and which are conjugated to subshifts of finite type. By Bowen's formula this is the same as finding such subsystems with pressure of the potential $-\dim_H(\Lambda)\log|Df|$ arbitrarily close to 0. Fernandez, Ugalde and Urias prove that for maps of the interval with a Markov partition by intervals this is always possible [4]. Their proof relies on combinatorial arguments. We provide here a proof based on the spectral approach, close in spirit to the work by Maume and Liverani [9]. Although more intricate, this approach allows us to consider easily higher dimensional conformal maps.

3.1. **Pressure for systems with small hole.** Let (X, T) be a subshift of finite type. Given an open set $H \subset X$, the hole, we set $Y = X \setminus H$ and denote by $Y^\infty = \cap_{m\geq 0} T^{-m}Y$ the compact forward invariant set of points never falling into the hole H. Our next result says that the pressure of Y^∞ tends to the pressure of the original system as the hole gets smaller. Notice that this is not a small perturbation for the potential, so that classical continuity of the pressure do not apply.

Proposition 7. *Let (X, T) be a topologically mixing subshift of finite type. Assume that $\varphi\colon X \to \mathbb{R}$ is a Hölder continuous potential. Let μ be the equilibrium measure of the potential φ. If H_n is a decreasing sequence of $\mathcal{F}_n$ measurable sets such that for some $c, \varepsilon > 0$*

$$\mu(H_n) \leq ce^{-\varepsilon n}$$

then $\lim_{n\to\infty} P(\varphi|(X \setminus H_n)^\infty) = P(\varphi|X)$.

Proof. The proof uses the characterization of the pressure by the logarithm of the spectral radius of the Perron-Frobenius operator associated to the system. Then we show that one can apply a perturbation result by Keller and Liverani [8] designed for these type of operators. Taking if necessary the new potential $\varphi - P(\varphi|X) + \psi \circ T - \psi$ we can assume that $P(\varphi|X) = 0$ and $L1 = 1$, where L is the Perron-Frobenius operator associated to (X, T, φ), i.e.

$$Lf(x) = \sum_{y, Ty=x} \mathrm{e}^{\varphi(y)} f(y).$$

Let $\mathcal{Z}_n$ be the partition of X by n-cylinders, and denote by $\mathcal{F}_n$ the finite σ-algebra generated by $\mathcal{Z}_n$. By the Gibbs property of μ there exists c_0 such that for any $x \in Z \in \mathcal{Z}_k$ we have $1/c_0 \leq \mu(Z)\mathrm{e}^{-\varphi_k(x)} < c_0$. We furthermore suppose that for any cylinders A, Z such that $AZ \overset{\text{def}}{=} A \cap T^{-k}Z \neq \varnothing$. we have $1/c_0 \leq \mu(A)\mu(Z)/\mu(AZ) \leq c_0$.

Given a function $f\colon X \to \mathbb{R}$ and a set $A \subset X$ let $\operatorname{osc}(f, A) = \sup_A f - \inf_A f$. Since φ is Hölder there exists c_1 and $\alpha < 1$ such that we have

$$\operatorname{var}_n(\varphi) \overset{\text{def}}{=} \sup_{Z\in\mathcal{Z}_n} \operatorname{osc}(\varphi, Z) \leq c_1\alpha^n, \quad \text{for any } n \geq 1. \tag{16}$$

Fix $\theta \in (1, \min(\alpha^{-1}, \mathrm{e}^{\varepsilon/2}))$. Define

$$\operatorname{osc}(f) = \sum_{n\geq 1} \theta^n \sum_{Z\in\mathcal{Z}_n} \operatorname{osc}(f, Z)\mu(Z).$$

Let $|f|_2 = (\int |f|^2 d\mu)^{1/2}$ and consider the Banach space

$$B = \{f\colon X \to \mathbb{R}\colon \|f\| \overset{\text{def}}{=} \operatorname{osc}(f) + |f|_2 < \infty\}.$$

We postpone the proofs of Lemmas 8 and 9 below to Section 4.

Lemma 8. *There exists constants c_3 and c_4 such that for any $k \geq 1$*

$$\operatorname{osc}(L^k f) \leq c_3\theta^{-k} \operatorname{osc}(f) + c_4|f|_2.$$

Let $H_n^k = H_n \cup T^{-1}H_n \cup \cdots T^{-k+1}H_n$. For simplicity we denote by A instead of 1_A the indicator function of a set A.

Lemma 9. *There exists a constant c_5 (depending on k) such that for any $N \in \mathbb{N}$ and $f \in B$,*

$$\operatorname{osc}(fH_N^k) \leq \operatorname{osc}(f) + c_5|f|_2.$$

For a set $Y \subset X$ we let L_Y be the perturbed operator, $L_Y f = L(Yf)$. Observe that $L_Y^k f = L^k(Y_k f)$, where $Y_k = Y \cap T^{-1}Y \cap \cdots T^{-k+1}Y$. Hence $L^k_{X\setminus H_N} f = L^k f - L^k(H_N^k f)$.

Lemma 10. *There exists k and c_6 such that for any $N \in \mathbb{N}$ and $f \in B$ we have*

$$\operatorname{osc}(L^k_{X\setminus H_N} f) \leq \frac{1}{2}\operatorname{osc}(f) + c_6|f|_2.$$

Proof of Lemma 10. Choose k such that $c_3\theta^{-k} \leq \frac{1}{4}$. We apply Lemma 8 and Lemma 9 successively

$$\begin{aligned}\operatorname{osc}(L^k_{X\setminus H_N} f) &\leq \operatorname{osc}(L^k f) + \operatorname{osc}(L^k(H_N^k f)) \\ &\leq \frac{1}{4}\operatorname{osc}(f) + c_4|f|_2 + \frac{1}{4}\operatorname{osc}(H_N^k f) + c_4|H_N^k f|_2 \\ &\leq \frac{1}{2}\operatorname{osc}(f) + (2c_4 + c_5)|f|_2.\end{aligned}$$

□

The following lemma express that the hole is a perturbation of the original system as far as operators from B to $L^2(\mu)$ are concerned.

Lemma 11. *For any $N \in \mathbb{N}$ and $f \in B$ with $\|f\| = 1$ we have*

$$|(L - L_{X\setminus H_N})f|_2 \leq c_0\mu(H_N)^{\frac{1}{4}}.$$

Proof of Lemma 11. Using Hölder inequality then the invariance of L we get

$$\int |L(H_N f)|^2 d\mu \leq \sup_X L|f| \int_{H_N} |f| d\mu \leq \mu(H_N)^{\frac{1}{2}} |f|_2 \sup_X L|f|.$$

Additionally, using again Hölder inequality, the supremum is bounded by

$$\sup_X L|f| \leq \sum_{Z\in\mathcal{Z}_1} c_0\mu(Z)\sup_Z |f| \leq \frac{c_0}{\theta}\operatorname{osc}(f) + c_0|f|_2.$$

□

It comes from the Folkolre Theorem that $1 = \mathrm{e}^{P(\varphi|X)}$ is a simple eigenvalue of L acting on B and that the rest of the spectrum is contained in a disc of radius $\rho < 1$. By Lemmas 8, 10 and 11 we can apply Keller and Liverani's theorem ([8], Corollary 1) with the sequence of operators $L_{X\setminus H_n}$ acting on B. This gives that this spectral figure will be conserved: for n sufficiently large, the spectrum of $L_{X\setminus H_n}$ on B outside the disc of radius ρ consists exactly in one simple eigenvalue λ_n, with eigenvector h_n, such that $\lambda_n \to 1$ and $h_n \xrightarrow{B} h$ (presumably λ_n and h_n are positive, but we won't use this fact).

Furthermore, the restriction of the eigenvector h_n to $(X \setminus H_n)^\infty$ is again an eigenvector of the Perron Frobenius operator $L_{(X\setminus H_n)^\infty}$ of the system $((X \setminus H_n)^\infty, T)$. Consequently the spectral radius of $L_{(X\setminus H_n)^\infty}$ on B is larger or equal to $|\lambda_n|$. Thus we get $P(\varphi|X) \geq P(\varphi|(X \setminus H_n)^\infty) \geq \log|\lambda_n|$, which proves the result. □

3.2. **Removing the boundary of the Markov partition.** The following proposition is of independent interest.

Proposition 12. *Let $\Lambda \subset \mathcal{M}$ be a conformal repeller of the $C^{1+\alpha}$ map $f\colon \mathcal{M} \to \mathcal{M}$. There exists subsets $\Lambda_n \subset \Lambda$ such that*
(i) (f, Λ_n) is conjugated to a subshift of finite type
(ii) $\lim_{n\to\infty} \dim_H \Lambda_n = \dim_H \Lambda$.

Proof. Let ξ be a generating Markov partition for the system (f, Λ) and (X, T) the corresponding subshift of finite type semi-conjugated via $\pi\colon X \to \Lambda$ to (f, Λ). We write $\varphi = -\dim_H(\Lambda)\log|Df|$.

Let $K = \pi^{-1}\partial\xi$ be the premiage of the boundary of the partition. The Markov property implies that $TK \subset K$. The compact invariant set K is strictly contained in X hence $P(\varphi|K) < 0$ and the $\mathcal{F}_n$ measurable sets $H_n = \mathcal{Z}_n(K)$ have a measure going exponentially to zero (see for example the appendix of [2] for a proof of these facts). By Proposition 7 for any $\epsilon > 0$ there exists $n(\epsilon)$ such that for any $n \geq n(\epsilon)$

$$P(\varphi|(X \setminus H_n)^\infty) > \epsilon \sup \varphi,$$

whence the subset $\Lambda_n \stackrel{\text{def}}{=} \pi(X \setminus H_n)^\infty \subset \Lambda$ has dimension

$$\dim_H(\Lambda_n) > (1 - \epsilon)\dim_H(\Lambda).$$

Moreover, (f, Λ_n) is conjugated to a subshift of finite type. □

We are now able to give a proof of our main theorem.

Proof of Theorem 1. By (i) in Proposition 12 we can apply Theorem 2 to (f, Λ_n). This provides us with a set $E_n(\alpha, \beta) \subset \Lambda_n$ (See (15)) such that $\dim E_n(\alpha, \beta) = \dim \Lambda_n$. By monotony of Hausdorff dimension we get,

$$\dim_H \Lambda \geq \dim_H E(\alpha, \beta) \geq \dim_H E_n(\alpha, \beta).$$

The result follows from (ii) in Proposition 12, taking the limit as $n \to \infty$. □

4. Proofs of the technical lemmas

Proof of Lemma 8. Let $\varphi_k = \varphi + \varphi \circ T + \cdots \varphi \circ T^{k-1}$. Note that using Equation (16) with $c_2 = c_1/(1-\alpha)$, for any integer n, k, we have $\mathrm{var}_{n+k}(\varphi_k) \leq c_2\alpha^n$. For a cylinder $A \in \mathcal{Z}_k$ and a point x we denote by

Ax the unique element of $A \cap T^{-k}x$, if any. If $Z \in \mathcal{Z}_n$ with $n \geq 1$ then

$$\begin{aligned}
\operatorname{osc}(L^k f, Z) &= \sup_{x,x' \in Z} L^k f(x) - L^k f(x') \\
&= \sup_{x,x' \in Z} \sum_{A \in \mathcal{Z}_k, AZ \neq \varnothing} e^{\varphi_k(Ax)} f(Ax) - e^{\varphi_k(Ax')} f(Ax') \\
&\leq \sup_{x,x' \in Z} \sum_A e^{\varphi_k(Ax)}(f(Ax) - f(Ax')) + \mathrm{e}^{\varphi_k(Ax)}(\mathrm{e}^{\operatorname{var}_{n+k}(\varphi_k)} - 1)\,|f|(Ax') \\
&\leq \sum_A c_0 \mu(A) \operatorname{osc}(f, AZ) + (\mathrm{e}^{\operatorname{var}_{n+k}(\varphi_k)} - 1) c_0 \mu(A) \sup_{AZ} |f| \\
&\leq c_0(1 + \mathrm{e}^{c_2}) \sum_A \mu(A) \operatorname{osc}(f, AZ) + \\
&\quad + c_0 \mathrm{e}^{c_2} c_2 \alpha^n \sum_A \mu(A) \mu(AZ)^{-1/2} \left(\int_{AZ} |f|^2 d\mu \right)^{1/2}.
\end{aligned}$$

Summing up over $Z \in \mathcal{Z}_n$ and then over n yields to

$$\begin{aligned}
\operatorname{osc}(L^k f) &\leq \sum_{n \geq 1} \theta^n \sum_{Z \in \mathcal{Z}_{n+k}} \Big[c_0^2 (1 + \mathrm{e}^{c_2}) \mu(Z) \operatorname{osc}(f, Z) + \\
&\quad + c_0^2 \mathrm{e}^{c_2} c_2 \alpha^n \mu(Z)^{1/2} \left(\int_Z |f|^2 d\mu \right)^{1/2} \Big] \\
&\leq c_3 \theta^{-k} \operatorname{osc}(f) + c_4 |f|_2.
\end{aligned}$$

For the last inequality we have set $c_3 = c_0^2(1+\mathrm{e}^{c_2})$ and $c_4 = c_0^2 \mathrm{e}^{c_2} c_2/(1-\alpha\theta)$ and use the Schwarz inequality. □

Proof of Lemma 9. Since H_N^k is $\mathcal{F}_{N+k}$ measurable we have

$$\begin{aligned}
\operatorname{osc}(f H_N^k) &= \sum_{n \geq 1} \theta^n \sum_{Z \in \mathcal{Z}_n} \operatorname{osc}(f H_N^k, Z) \mu(Z) \\
&\leq \sum_{n=1}^{N+k-1} \theta^n \sum_{Z \in \mathcal{Z}_n, Z \cap H_N^k \neq \varnothing} \mu(Z) \sup_Z |f| + \sum_{n \geq N+k} \theta^n \sum_{Z \in \mathcal{Z}_n} \operatorname{osc}(f, Z) \mu(Z) \\
&\leq \operatorname{osc}(f) + \sum_{n=1}^{N+k-1} \theta^n \sum_{Z \in \mathcal{Z}_n, Z \cap H_N^k \neq \varnothing} \mu(Z)^{1/2} \left(\int_Z |f|^2 d\mu \right)^{1/2} \\
&\leq \operatorname{osc}(f) + \sum_{n=1}^{N+k-1} \theta^n \mu(\mathcal{Z}_n(H_N^k))^{1/2} |f|_2.
\end{aligned}$$

For the last inequality we used Schwarz inequality and $\mathcal{Z}_n(H_N^k)$ denotes the union of elements of $\mathcal{Z}_n$ intersecting H_N^k. For any $n \leq N + k - 1$ we have

$$\mathcal{Z}_n(H_N^k) \subset \cup_{j=0}^{k-1} T^{-j} \mathcal{Z}_{n-j}(H_N) \subset \cup_{j=0}^{k-1} T^{-j} H_{n-k}.$$

The result follows from the invariance of the measure by taking

$$c_5 = \frac{\sqrt{ck\mathrm{e}^{\varepsilon k}}}{1 - \theta \mathrm{e}^{-\frac{\varepsilon}{2}}} \geq \sup_N \sum_{n=1}^{N+k-1} \theta^n (kc\mathrm{e}^{-\varepsilon(n-k)})^{1/2}.$$

□

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