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PRODUCT STRUCTURE OF POINCARÉ RECURRENCE

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Abstract. We provide new non-trivial quantitative information on the behavior of Poincaré recurrence. In particular we establish the almost everywhere coincidence of the recurrence rate and of the pointwise dimension for a large class of repellers, including repellers without finite Markov partitions.

Using this information, we are able to show that for locally maximal hyperbolic sets the recurrence rate possesses a certain local product structure, which closely imitates the product structure provided by the families of local stable and unstable manifolds, as well as the almost product structure of hyperbolic measures.

1. Introduction

It is well known that the classical Poincaré recurrence theorem only gives information of *qualitative* nature. On the other hand it is clearly a matter of intrinsic difficulty and not of lack interest that little is know concerning the *quantitative* nature of recurrence. This paper contributes to the solution of this problem and has two main objectives:

1. we want to provide new non-trivial quantitative information on the behavior of recurrence, establishing the almost everywhere coincidence of recurrence rate and of pointwise dimension for a large class of repellers, possibly without finite Markov partitions;
2. we want to show that for hyperbolic sets the recurrence rate possesses a product structure, which closely imitates the product structure provided by the families of local stable and unstable manifolds, and the almost product structure of hyperbolic measures (see below for definitions and for a detailed description).

Instead of formulating general statements at this point, we choose to illustrate our results with particular cases.

Fix $d \in \mathbb{N}$ and consider a $d \times d$ matrix A with integer entries. It induces a *toral endomorphism* $T_A \colon \mathbb{T}^d \to \mathbb{T}^d$ of the d-dimensional torus by left-multiplication by A. The map T_A is invertible if and only if $|\det A| = 1$ and in this case it is also called a *toral automorphism*.

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Theorem 1. *If T_A is a toral endomorphism of $\mathbb{T}^d$ induced by a matrix A with all eigenvalues outside the unit disk, then*

$$\lim_{r\to 0} \frac{\log \inf\{n \in \mathbb{N} \colon \|{T_A}^n x - x\| < r\}}{-\log r} = d \tag{1}$$

for Lebesgue-almost every $x \in \mathbb{T}^d$.

This statement shows that for Lebesgue-almost every point $x \in \mathbb{T}^d$ the first time that the positive orbit of x under T_A returns to the open ball $B(x,r) \subset \mathbb{T}^d$ of radius r centered at x is approximately r^{-d}, and we shall also write

$$\inf\{n \in \mathbb{N} \colon {T_A}^n x \in B(x,r)\} \sim \frac{1}{r^d} \text{ as } r \to 0.$$

When $d = 1$, the endomorphism $T_A \colon [0,1] \to [0,1]$ can be written in the form $T_A x = mx \pmod 1$ for some integer $m \in \mathbb{N} \setminus \{1\}$. Let $x = 0.x_1x_2\cdots$ be the base-m representation of the point $x \in [0,1]$ (note that it is uniquely defined except on a countable subset of $[0,1]$). In this case the statement in Theorem 1 can be reformulated in the following way: for Lebesgue-almost every $x \in [0,1]$ we have

$$\inf\{n \in \mathbb{N} \colon |0.x_n x_{n+1} \cdots - 0.x_1 x_2 \cdots| < r\} \sim \frac{1}{r} \text{ as } r \to 0.$$

We now consider another example.

Theorem 2. *If $T \colon (0,1) \to (0,1)$ is the Gauss map, that is, $T(x) = 1/x \pmod 1$ for each $x \in (0,1)$, then*

$$\lim_{r\to 0} \frac{\log \inf\{n \in \mathbb{N} : |T^n x - x| < r\}}{-\log r} = 1 \tag{2}$$

for Lebesgue-almost every $x \in (0,1)$.

Writing each number $x \in (0,1)$ as a continued fraction

$$x = [m_1, m_2, m_3, \ldots] \stackrel{\text{def}}{=} \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{m_3 + \cdots}}},$$

with $m_i = m_i(x) \in \mathbb{N}$ for each i (note that this representation is unique except on a countable subset of $(0,1)$), one can reformulate Theorem 2 in the following way: for Lebesgue-almost every $x \in (0,1)$ we have

$$\inf\{n \in \mathbb{N} : |[m_n, m_{n+1}, \ldots] - [m_1, m_2, \ldots]| < r\} \sim \frac{1}{r} \text{ as } r \to 0.$$

Theorems 1 and 2 are consequences of the more general results in Section 2 below, and provide new non-trivial information about the quantitative behavior of recurrence.

The study of the quantitative behavior of recurrence started with the work of Ornstein and Weiss [7], closely followed by the work of Boshernitzan [3]. In [7] the authors establish an identity similar to those in Theorems 1 and 2 (see (1) and (2)) in the case of symbolic dynamics, and thus for the corresponding symbolic metric (see also Remark 2 in Section 3.2 below). On the other hand, the paper [3] is closer in spirit to the present paper, in the sense that Boshernitzan also considered arbitrary metric spaces. More precisely, he established a general inequality between quantities related to those in the right and left-hand sides of the identities in (1) and (2) (although the result in [3] is formulated differently it is shown in [2] how it can be rephrased in this manner). We showed in [2] that this inequality is strict in general, and provided a sharper inequality which becomes an identity for large classes of maps. In particular, for sufficiently regular hyperbolic diffeomorphisms we established an identity which is a version of the one obtained by Ornstein and Weiss in [7] in the special case of symbolic dynamics. The present paper is a further contribution to the study of the quantitative behavior of recurrence.

We now consider a $C^{1+\alpha}$ diffeomorphism f with a compact locally maximal hyperbolic set Λ. We want to show that the return time possesses a certain *local product structure*. More precisely, we shall show that the return time of the orbit of a point $x \in \Lambda$ under f to the ball $B(x, r)$ is approximately the product of the return times to the r-neighborhoods of the stable and unstable manifolds of x. This includes:

1. the separate study of the return times in the stable and unstable directions;
2. the study of the relation between these numbers and the return time to a small neighborhood of a given point.

The first point can be addressed in the following manner. The diffeomorphism f can be modeled by two repellers for certain "stable" and "unstable" maps (this is made precise in Section 4), respectively in the stable and unstable directions. For these repellers one can establish appropriate versions of the results in Theorems 1 and 2.

In order to formulate these results, we consider the lower and upper stable recurrence rates $\underline{R}^s(x)$ and $\overline{R}^s(x)$, and the lower and upper unstable recurrence rates $\underline{R}^u(x)$ and $\overline{R}^u(x)$ of a point x (see (16) and (17) in Section 3 for the definition). Given a probability measure μ on Λ, we also consider the families of conditional measures μ_x^s and μ_x^u associated respectively to the partitions ξ^s and ξ^u into local stable and unstable manifolds constructed by Ledrappier and Young in [6] (see Section 3 for details). We can now formulate a rigorous statement (see also Theorem 9 below).

Theorem 3. *For a topologically mixing $C^{1+\alpha}$ diffeomorphism on a compact locally maximal hyperbolic set Λ, and an equilibrium measure μ of a Hölder continuous function, we have*

$$R^s(x) \stackrel{\text{def}}{=} \underline{R}^s(x) = \overline{R}^s(x) = \lim_{r\to 0} \frac{\log \mu_x^s(B^s(x,r))}{\log r} \tag{3}$$

and

$$R^u(x) \stackrel{\text{def}}{=} \underline{R}^u(x) = \overline{R}^u(x) = \lim_{r\to 0} \frac{\log \mu_x^u(B^u(x,r))}{\log r} \tag{4}$$

for μ-almost every $x \in \Lambda$.

The limits in the right-hand side of the identities in (3) and (4) have been shown to exist by Ledrappier and Young in [6]. We emphasize that the identities in Theorem 3 relate quantities of very different nature. While the numbers $R^s(x)$ and $R^u(x)$ are essentially quantities of geometric nature and are independent of the measure μ, the limits in (3) and (4) are essentially of measure-theoretical nature.

It follows from work of Barreira, Pesin and Schmeling [1] on the product structure of hyperbolic measures that

$$\lim_{r\to 0}\frac{\log\mu_x^s(B^s(x,r))}{\log r}+\lim_{r\to 0}\frac{\log\mu_x^u(B^u(x,r))}{\log r}=\lim_{r\to 0}\frac{\log\mu(B(x,r))}{\log r} \tag{5}$$

for μ-almost every $x\in\Lambda$. Furthermore, it follows from work of Barreira and Saussol [2] that

$$\lim_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}=\lim_{r\to 0}\frac{\log\inf\{n\in\mathbb{N}: d(f^n x,x)<r\}}{-\log r} \tag{6}$$

for μ-almost every $x\in\Lambda$. Combining Theorem 3 with the identities in (5) and (6) we obtain the remarkable identity

$$R^s(x)+R^u(x)=\lim_{r\to 0}\frac{\log\inf\{n\in\mathbb{N}: d(f^n x,x)<r\}}{-\log r}$$

for μ-almost every $x\in\Lambda$. This identity implies that the return time to a ball is approximately equal to the product of the return times into the stable and the unstable balls of the same size. The precise formulation is given in Section 3 (see Theorem 7 below).

The structure of the paper is as follows. In Section 2 we study the recurrence rate for repellers. In particular we consider a class of repellers without finite Markov partitions (as in Theorem 2 above). In Section 3 we study the product structure of Poincaré recurrence on locally maximal hyperbolic sets of $C^{1+\alpha}$ diffeomorphisms. The approach to this problem is effected through a careful study of certain induced maps on the stable and unstable directions provided by the families of invariant manifolds; see Section 4. These induced maps are special cases of the repellers considered in Section 2, and thus the results in Sections 2 and 3 are strongly related. The proofs of the results in Sections 1–3 are collected in Section 5.

In Appendix A we establish results of independent interest, and which are crucial to the proofs. In particular we show that the μ-measure of the ε-neighborhood of the boundary $\partial\mathcal{R}$ of a Markov partition $\mathcal{R}$ (for repellers or locally maximal hyperbolic sets, and arbitrary Gibbs measures) is at most polynomial in ε, i.e. there exist constants $c>0$ and $\nu>0$ such that if $\varepsilon>0$ then

$$\mu(\{x\in\Lambda\colon d(x,\partial\mathcal{R})<\varepsilon\})\le c\varepsilon^\nu. \tag{7}$$

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2. Recurrence rate for repellers

2.1. **Repellers of smooth maps.** Let $T\colon M\to M$ be a C^1 map of a smooth Riemannian manifold. Consider a compact f-invariant set $X\subset M$. We say that T is *expanding* on X, and that X is a *repeller* of T if there exist constants $c > 0$ and $\beta > 1$ such that $\|d_x T^n u\| \geq c\beta^n \|u\|$ for all $x \in X$, $u \in T_x M$, and $n \in \mathbb{N}$.

We define the *return time* of a point $x \in M$ into the open ball $B(x, r)$ by

$$\tau_r(x) \stackrel{\text{def}}{=} \inf\{n \in \mathbb{N} : T^n x \in B(x,r)\} = \inf\{n \in \mathbb{N} : d(T^n x, x) < r\}.$$

The *lower* and *upper recurrence rates* of x are defined by

$$\underline{R}(x) = \varliminf_{r \to 0} \frac{\log \tau_r(x)}{-\log r} \quad \text{and} \quad \overline{R}(x) = \varlimsup_{r \to 0} \frac{\log \tau_r(x)}{-\log r}. \tag{8}$$

Given a probability measure μ on X, the *lower* and *upper pointwise dimensions* of μ at a point $x \in X$ are defined by

$$\underline{d}_\mu(x) = \varliminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \varlimsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

The following statement relates the recurrence rates with the pointwise dimensions.

Theorem 4. *Let X be a repeller of a topologically mixing $C^{1+\alpha}$ expanding map, and μ an equilibrium measure of a Hölder continuous function. Then*

$$\underline{R}(x) = \overline{R}(x) = \underline{d}_\mu(x) = \overline{d}_\mu(x) \tag{9}$$

for μ-almost every $x \in X$.

In [9], Schmeling and Troubetzkoy proved that if μ is a probability measure invariant under a $C^{1+\alpha}$ expanding map on X, then $\underline{d}_\mu(x) = \overline{d}_\mu(x)$ for μ-almost every $x \in X$.

We remark that Theorem 4 relates quantities of very different nature. In particular, the numbers $\underline{R}(x)$ and $\overline{R}(x)$ do not depend on the measure, while the numbers $\underline{d}_\mu(x)$ and $\overline{d}_\mu(x)$ do not depend on the map.

2.2. **Generalized repellers.** In this section we present a large class of generalized repellers for which one can establish a version of Theorem 4.

Let $g \colon W \to W$ be a Borel-measurable transformation on the metric space W, and μ a g-invariant probability measure on W. Recall that the entropy of a finite or countable partition $\mathcal{Z}$ of W by measurable sets is given by

$$H_\mu(\mathcal{Z}) \stackrel{\text{def}}{=} -\sum_{Z \in \mathcal{Z}} \mu(Z) \log \mu(Z).$$

For each $n \in \mathbb{N}$ we define the new partition $\mathcal{Z}_n = \bigvee_{k=0}^{n-1} g^{-k}\mathcal{Z}$. Given $x \in W$ and $n \in \mathbb{N}$ we denote by $\mathcal{Z}_n(x) \in \mathcal{Z}_n$ the unique (mod 0) element of $\mathcal{Z}_n$ which contains the point x.

Theorem 5. *Let g be a Borel-measurable transformation of a set $W \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$, μ an ergodic g-invariant non-atomic probability measure on W, and $\mathcal{Z}$ a finite or countable partition of W by measurable sets with $H_\mu(\mathcal{Z}) < \infty$. Assume that:*

1. *there exists $\kappa > 1$ such that if n, $m \in \mathbb{N}$ and $x \in W$, then*

$$\mu(\mathcal{Z}_{n+m}(x)) \leq \kappa \mu(\mathcal{Z}_n(x)) \mu(\mathcal{Z}_m(g^n x)); \tag{10}$$

2. *there exists* $\lambda > 0$ *such that for each sufficiently large* $n \in \mathbb{N}$ *we have*

$$\sup_{Z \in \mathcal{Z}_n} \operatorname{diam} Z < e^{-\lambda n}; \tag{11}$$

3. *for* μ*-almost every* $x \in X$ *there exists* $\gamma > 0$ *such that* $B(x, e^{-\gamma n}) \subset \mathcal{Z}_n(x)$ *for all sufficiently large* n.

Then for μ*-almost every* $x \in W$, *we have*

$$\underline{R}(x) = \underline{d}_\mu(x) \quad \text{and} \quad \overline{R}(x) = \overline{d}_\mu(x). \tag{12}$$

The hypotheses 2 and 3 in the theorem concern respectively the size and "inner size" of the elements of the partitions $\mathcal{Z}_n$. The exponential behavior justifies the expression "generalized repeller" when referring to the set W in Theorem 5. On the other hand hypothesis 1 is a one-sided Gibbs-type inequality for the measure μ with respect to the partition $\mathcal{Z}$. It is related to the occurrence of bounded distortion for the map g. By the Whitney embedding theorem one can also apply Theorem 5 when W is a subset of a smooth manifold.

Dynamical systems satisfying the hypotheses of Theorem 5 include:

1. repellers of $C^{1+\alpha}$ expanding maps together with equilibrium measures of Hölder continuous functions (see Section 2.1 and the proof of Theorem 4);
2. the Gauss map endowed with its unique absolutely continuous invariant probability measure μ given by

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{dx}{1+x} \tag{13}$$

(see the proof of Theorem 2);

3. induced maps in the stable and unstable directions for locally maximal hyperbolic sets of $C^{1+\alpha}$ diffeomorphisms together with equilibrium measures of Hölder continuous functions (see Section 4 and the proof of Theorem 10).

In particular, Theorems 2, 4, and 10 provide applications of Theorem 5 to three different situations. One can also apply Theorem 5 to several transformations obtained from the induction of non-uniformly expanding maps, such as the Pommeau–Manneville map.

2.3. **Dimension of measures.** In this section we relate the previous results with the dimension of measures invariant under the dynamical system.

We briefly recall the notion of Hausdorff dimension. Let X be a separable metric space. Given a subset $Z \subset X$ and a real number $\alpha \geq 0$, we define the *α-dimensional Hausdorff measure* of Z by

$$m_\alpha(Z) = \lim_{\delta \to 0} \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} (\operatorname{diam} U)^\alpha,$$

where the infimum is taken over all finite or countable covers $\mathcal{U}$ of Z by sets of diameter at most δ. The *Hausdorff dimension* of Z is defined by

$$\dim_H Z = \inf\{\alpha : m_\alpha(Z) = 0\}.$$

We also define the *lower* and *upper box dimensions* of Z by

$$\underline{\dim}_B Z = \varliminf_{r\to 0} \frac{\log N(Z,r)}{-\log r} \quad \text{and} \quad \overline{\dim}_B Z = \varlimsup_{r\to 0} \frac{\log N(Z,r)}{-\log r},$$

where $N(Z,r)$ denotes the smallest number of balls of diameter r needed to cover Z. It is easy to verify that

$$\dim_H Z \le \underline{\dim}_B Z \le \overline{\dim}_B Z.$$

The *Hausdorff dimension* of a probability measure μ on X is given by

$$\dim_H \mu = \inf\{\dim_H Z : \mu(Z) = 1\},$$

while the *lower* and *upper box dimensions* of μ are defined by

$$\underline{\dim}_B \mu = \lim_{\delta\to 0} \inf\{\underline{\dim}_B Z : \mu(Z) \ge 1-\delta\}$$

and

$$\overline{\dim}_B \mu = \lim_{\delta\to 0} \inf\{\overline{\dim}_B Z : \mu(Z) \ge 1-\delta\}.$$

One can easily verify that

$$\dim_H \mu \le \underline{\dim}_B \mu \le \overline{\dim}_B \mu.$$

Under the assumptions of Theorem 4, the measure μ is ergodic and thus (since the quantities in (9) are T-invariant μ-almost everywhere) there exists a constant r_μ such that

$$\underline{R}(x) = \overline{R}(x) = \underline{d}_\mu(x) = \overline{d}_\mu(x) = r_\mu$$

for μ-almost every $x \in X$. The following statement is now an immediate consequence of a criterion of Young in [10].

Proposition 6. *If X is a repeller of a topologically mixing $C^{1+\alpha}$ expanding map, and μ is an equilibrium measure of a Hölder continuous function, then*

$$\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = r_\mu.$$

Other quantities of dimensional nature which also coincide with r_μ are described in [1].

Recall that a differentiable map T is said to be *conformal* on X if $d_x T$ is a multiple of an isometry for every $x \in X$. For example, this happens whenever X is an interval or T is a holomorphic map. For conformal expanding maps, we have

$$\underline{d}_\mu(x) = \overline{d}_\mu(x) = h_\mu(T)/\chi(x)$$

for μ-almost every $x \in X$, where $h_\mu(T)$ the μ-entropy of T and $\chi(x)$ is the Lyapunov exponent at x. By Birkhoff's ergodic theorem, we have

$$\chi(x) \stackrel{\text{def}}{=} \varlimsup_{n\to\infty} \frac{1}{n} \log\|d_x T^n\| = \varlimsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log\|d_{T^k x} T\| = \int_X \log\|dT\|\, d\mu$$

for μ-almost every $x \in X$, and thus (for conformal maps) we obtain

$$\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = \frac{h_\mu(T)}{\int_X \log\|dT\|\, d\mu} = r_\mu.$$

3. Product structure of return times

3.1. **Product structure.** Let f be a topologically mixing $C^{1+\alpha}$ diffeomorphism on a smooth Riemannian manifold M, and $\Lambda \subset M$ a compact locally maximal hyperbolic set for f.

To each point $x \in \Lambda$ one can associate its *stable* and *unstable (global) manifolds* defined by

$$W^s(x) = \{y \in M \colon d(f^n x, f^n y) \to 0 \text{ as } n \to +\infty\},$$
$$W^u(x) = \{y \in M \colon d(f^n x, f^n y) \to 0 \text{ as } n \to -\infty\},$$

where d denotes the Riemannian distance on M. We also denote by d^s and d^u the distances induced by d respectively on the manifolds $W^s(x)$ and $W^u(x)$.

Given $\varepsilon > 0$, let $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ be respectively the connected components of $W^s(x) \cap B(x,\varepsilon)$ and $W^u(x) \cap B(x,\varepsilon)$ containing x. There exists $\varepsilon > 0$ (which does not depend on $x \in \Lambda$) such that $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ are embedded manifolds, each dividing $B(x,\varepsilon)$ into two connected components. We call $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ respectively the *local stable* and *unstable manifolds* of $x \in \Lambda$ (of size ε).

Furthermore, there exists $\delta = \delta(\varepsilon) > 0$ such that for any x, $y \in \Lambda$ with $d(x,y) \le \delta$ the intersection $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ contains exactly one point. One can show that the map

$$[\cdot,\cdot]\colon \{(x,y) \in \Lambda \times \Lambda : d(x,y) \le \delta\} \to M$$

defined by

$$(x,y) \mapsto [x,y] \stackrel{\text{def}}{=} W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$$

is a (local) Hölder homeomorphism. For each $\rho \le \delta$ we define the *stable* and *unstable return times* (see Figure 1) of $x \in \Lambda$ into the ball of radius r respectively by

$$\tau^s_r(x,\rho) \stackrel{\text{def}}{=} \inf\{n \in \mathbb{N} : d(f^{-n}x, x) \le \rho \text{ and } d^s([x, f^{-n}x], x) < r\},$$
$$\tau^u_r(x,\rho) \stackrel{\text{def}}{=} \inf\{n \in \mathbb{N} : d(f^{n}x, x) \le \rho \text{ and } d^u([f^n x, x], x) < r\}.$$

We note that the stable return time for f is equal to the unstable return time for f^{-1}.

Set

$$\underline{R}^s(x,\rho) \stackrel{\text{def}}{=} \varliminf_{r\to 0} \frac{\log \tau^s_r(x,\rho)}{-\log r} \quad \text{and} \quad \overline{R}^s(x,\rho) \stackrel{\text{def}}{=} \varlimsup_{r\to 0} \frac{\log \tau^s_r(x,\rho)}{-\log r}, \tag{14}$$

$$\underline{R}^u(x,\rho) \stackrel{\text{def}}{=} \varliminf_{r\to 0} \frac{\log \tau^u_r(x,\rho)}{-\log r} \quad \text{and} \quad \overline{R}^u(x,\rho) \stackrel{\text{def}}{=} \varlimsup_{r\to 0} \frac{\log \tau^u_r(x,\rho)}{-\log r}. \tag{15}$$

We define the *lower* and *upper stable recurrence rates* of the point $x \in \Lambda$ by

$$\underline{R}^s(x) \stackrel{\text{def}}{=} \lim_{\rho\to 0} \underline{R}^s(x,\rho) \quad \text{and} \quad \overline{R}^s(x) \stackrel{\text{def}}{=} \lim_{\rho\to 0} \overline{R}^s(x,\rho), \tag{16}$$

and the *lower* and *upper unstable recurrence rates* of the point $x \in \Lambda$ by

$$\underline{R}^u(x) \stackrel{\text{def}}{=} \lim_{\rho\to 0} \underline{R}^u(x,\rho) \quad \text{and} \quad \overline{R}^u(x) \stackrel{\text{def}}{=} \lim_{\rho\to 0} \overline{R}^u(x,\rho). \tag{17}$$

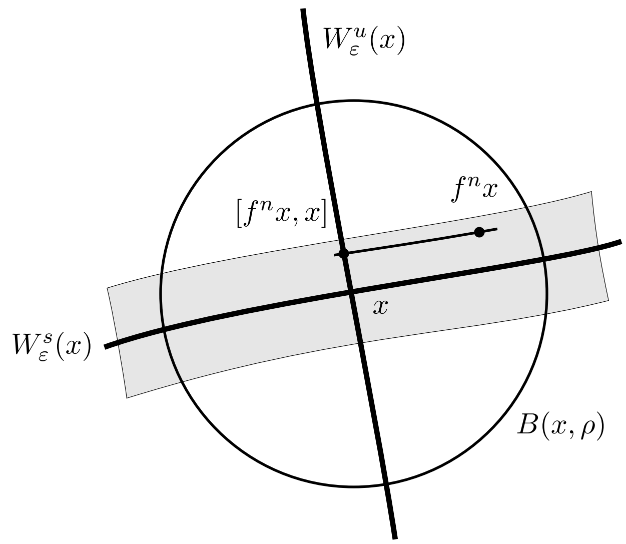

FIGURE 1. Definition of the unstable return time; the grey zone is the set of points whose d^u-distance to $W^s_\varepsilon(x)$ is at most r.

Since the functions $\rho \mapsto \tau^s_r(x,\rho)$ and $\rho \mapsto \tau^u_r(x,\rho)$ are non-decreasing, the limits in (16) and (17) are well-defined. One can easily show that the functions $\underline{R}^s$, $\overline{R}^s$, $\underline{R}^u$, and $\overline{R}^u$ are f-invariant on Λ. Whenever $\underline{R}^s(x) = \overline{R}^s(x)$ we shall denote the common value by $R^s(x)$, and whenever $\underline{R}^u(x) = \overline{R}^u(x)$ we shall denote the common value by $R^u(x)$.

When Λ is a compact locally maximal hyperbolic set for a topologically mixing $C^{1+\alpha}$ diffeomorphism, and μ is an equilibrium measure of a Hölder continuous function, we can show (see Theorem 9 below) that for μ-almost every $x \in \Lambda$ each of the numbers in (14) and (15) are independent of ρ for all sufficiently small ρ.

We also consider the *return time* of the point $x \in \Lambda$ into the open ball $B(x,r)$ given by

$$\tau_r(x) = \inf\{n \in \mathbb{N} : d(f^n x, x) < r\},$$

and define its *lower* and *upper recurrence rates* $\underline{R}(x)$ and $\overline{R}(x)$ as in (8). One can easily show that the functions $\underline{R}$ and $\overline{R}$ are f-invariant on Λ. Whenever $\underline{R}(x) = \overline{R}(x)$ we shall denote the common value by $R(x)$.

We now state the main result of this section.

Theorem 7. *Let Λ be a compact locally maximal hyperbolic set of a topologically mixing $C^{1+\alpha}$ diffeomorphism, and μ an equilibrium measure of a Hölder continuous function. For μ-almost every $x \in \Lambda$ the following properties hold:*

1. *the recurrence rate is equal to the sum of the stable and unstable recurrence rates, i.e.*

$$R(x) = R^s(x) + R^u(x);$$

2. *there exists $\rho(x) > 0$ such that for each $\rho < \rho(x)$ and each $\varepsilon > 0$ there exists $r(x, \rho, \varepsilon) > 0$ such that if $r < r(x, \rho, \varepsilon)$ then*

$$r^{\varepsilon} < \frac{\tau_r^s(x, \rho) \cdot \tau_r^u(x, \rho)}{\tau_r(x)} < r^{-\varepsilon}.$$

Theorem 7 reveals a local product structure for return times and a local product structure for recurrence rates. Theorem 7 is an application of Theorem 9 below and of work of Barreira and Saussol in [2].

The following is an application of Theorem 7.

Theorem 8. *For a topologically mixing volume-preserving $C^{1+\alpha}$ Anosov diffeomorphism on a compact d-dimensional manifold M, and for Lebesgue-almost every $x \in M$ there exists $\rho(x) > 0$ such that for each $\rho < \rho(x)$ and each $\varepsilon > 0$ there exists $r(x, \rho, \varepsilon) > 0$ such that if $r < r(x, \rho, \varepsilon)$ then*

$$r^{d+\varepsilon} < \tau_r^s(x, \rho) \cdot \tau_r^u(x, \rho) < r^{d-\varepsilon}.$$

Furthermore, under the hypotheses of Theorem 8, we have

$$R(x) = d, \quad R^s(x) = d_s, \quad R^u(x) = d_u$$

for Lebesgue-almost every $x \in M$, where d_s and d_u denote respectively the dimensions of the stable and unstable manifolds. In particular, Theorem 8 readily applies to hyperbolic toral automorphisms.

3.2. **Stable and unstable recurrence rates.** Let now μ be a Borel probability measure invariant under a $C^{1+\alpha}$ diffeomorphism on the manifold M. In [6], Ledrappier and Young constructed two measurable partitions ξ^s and ξ^u of M such that for μ-almost every $x \in M$:

1. $\xi^s(x) \subset W_\varepsilon^s(x)$ and $\xi^u(x) \subset W_\varepsilon^u(x)$;
2. there exists $\gamma = \gamma(x) > 0$ such that

$$\xi^s(x) \supset W_\varepsilon^s(x) \cap B(x, \gamma) \quad \text{and} \quad \xi^u(x) \supset W_\varepsilon^s(x) \cap B(x, \gamma).$$

We denote by μ_x^s and μ_x^u the conditional measures associated respectively to the partitions ξ^s and ξ^u. Recall that any measurable partition ξ of M has associated a family of conditional measures: for μ-almost every $x \in M$ there exists a probability measure μ_x defined on the unique (mod 0) element $\xi(x)$ of ξ containing x. The conditional measures are characterized completely by the following property: if $\mathcal{B}_\xi$ is the σ-subalgebra of the Borel σ-algebra generated by unions of elements of ξ then for each Borel set $A \subset M$, the function $x \mapsto \mu_x^s(A \cap \xi(x))$ is $\mathcal{B}_\xi$-measurable and

$$\mu(A) = \int_A \mu_x^s(A \cap \xi(x)) \, d\mu.$$

We represent by $B^s(x, r) \subset W_\varepsilon^s(x)$ and $B^u(x, r) \subset W_\varepsilon^u(x)$ the open balls centered at x with radius r with respect to the distances d^s and d^u. It follows from work of Ledrappier and Young in [6] that if μ is an ergodic f-invariant probability measure supported on a locally maximal hyperbolic set Λ of a $C^{1+\alpha}$ diffeomorphism, then there exist constants d_μ^s and d_μ^u such that

$$\lim_{r \to 0} \frac{\log \mu_x^s(B^s(x, r))}{\log r} = d_\mu^s \quad \text{and} \quad \lim_{r \to 0} \frac{\log \mu_x^s(B^u(x, r))}{\log r} = d_\mu^u$$

for μ-almost every $x \in \Lambda$. The following statement relates these quantities with the stable and unstable recurrence rates.

Theorem 9. *Let Λ be a compact locally maximal hyperbolic set of a topologically mixing $C^{1+\alpha}$ diffeomorphism, and μ an equilibrium measure of a Hölder continuous function. For μ-almost every $x \in \Lambda$ there exists $\rho(x) > 0$ such that if $\rho < \rho(x)$ then*

$$R^s(x) = \underline{R}^s(x,\rho) = \overline{R}^s(x,\rho) = d_\mu^s$$

and

$$R^u(x) = \underline{R}^u(x,\rho) = \overline{R}^u(x,\rho) = d_\mu^u.$$

Theorem 7 above provides a non-trivial application of Theorem 9.

Remark 1. Theorem 7 is a counterpart for Poincaré recurrence of work by Barreira, Pesin, and Schmeling in [1] for pointwise dimension. Both works establish a local product structure, although for quite different quantities. Theorem 9 is a counterpart for Poincaré recurrence of work by Ledrappier and Young in [6] for the pointwise dimension.

Remark 2. The previous results should be compared with work of Ornstein and Weiss for the entropy. They showed in [7] that if $\sigma^+ \colon \Sigma^+ \to \Sigma^+$ is a *one-sided* subshift and μ^+ is a σ^+-invariant ergodic probability measure on Σ^+, then

$$\lim_{k\to\infty} \frac{\log \inf\{n \in \mathbb{N} : (i_{n+1} \cdots i_{n+k}) = (i_1 \cdots i_k)\}}{k} = h_{\mu^+}(\sigma) \tag{18}$$

for μ^+-almost every $(i_1 i_2 \cdots) \in \Sigma^+$. They also showed in [7] that if $\sigma \colon \Sigma \to \Sigma$ is a *two-sided* subshift and μ is a σ-invariant ergodic probability measure on Σ, then

$$\lim_{k\to\infty} \frac{\log \inf\{n \in \mathbb{N} : (i_{n-k} \cdots i_{n+k}) = (i_{-k} \cdots i_k)\}}{2k+1} = h_\mu(\sigma) \tag{19}$$

for μ-almost every $(\cdots i_{-1} i_0 i_1 \cdots) \in \Sigma$.

Given a two-sided shift $\sigma \colon \Sigma \to \Sigma$ it has naturally associated two one-sided shifts $\sigma^+ \colon \Sigma^+ \to \Sigma^+$ and $\sigma^- \colon \Sigma^- \to \Sigma^-$ (respectively related with the future and with the past). Furthermore, any σ-invariant measure on Σ induces a σ^+-invariant measure μ^+ on Σ^+ and a σ^--invariant measure μ^- on Σ^-, such that

$$h_{\mu^+}(\sigma^+) = h_{\mu^-}(\sigma^-) = h_\mu(\sigma).$$

For each $\omega = (\cdots i_{-1} i_0 i_1 \cdots) \in \Sigma$ and $k \in \mathbb{N}$ we set

$$\begin{aligned}
\tau_k^+(\omega) &= \inf\{n \in \mathbb{N} : (i_{n+1} \cdots i_{n+k}) = (i_1 \cdots i_k)\},\\
\tau_k^-(\omega) &= \inf\{n \in \mathbb{N} : (i_{-n-k} \cdots i_{-n-1}) = (i_{-k} \cdots i_{-1})\},\\
\tau_k(\omega) &= \inf\{n \in \mathbb{N} : (i_{n-k} \cdots i_{n+k}) = (i_{-k} \cdots i_k)\}.
\end{aligned}$$

It follows from (18) and (19) that for μ-almost every $\omega \in \Sigma$, given $\varepsilon > 0$ if $k \in \mathbb{N}$ is sufficiently large then

$$\mathrm{e}^{-k\varepsilon} \le \frac{\tau_k^+(\omega)\tau_k^-(\omega)}{\tau_k(\omega)} \le \mathrm{e}^{k\varepsilon}.$$

Theorems 7 and 9 can be considered versions of these statements in the case of dimension.

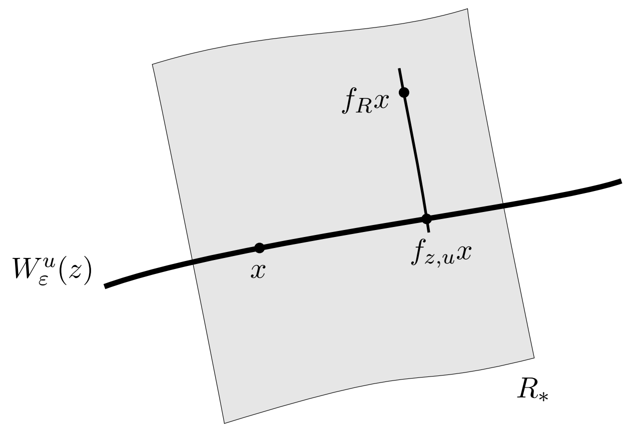

FIGURE 2. Construction of the induced map $f_{z,u}$ on $W^u_\varepsilon(z) \cap R_*$.

4. Induced dynamics

In order to help establishing Theorem 9 (to be proven in Section 5.1 below), we shall show here that one can relate the unstable recurrence rate to the return rate of a certain "unstable map" which satisfies the hypotheses of Theorem 5. This "unstable map" is induced by the dynamics on the unstable direction of each element of a fixed Markov partition. A similar study can be effected for the stable recurrence rate. This allows one to see Theorem 9 as a further non-trivial application of Theorem 5.

Let f and Λ be as in Section 3.1. Let now $\mathcal{R} = \{R_1, \dots, R_\ell\}$ be a Markov partition of the compact locally maximal hyperbolic set Λ. We recall that $\mathcal{R}$ satisfies the following properties (with the interior computed with respect to the induced topology on Λ):

1. if $R \in \mathcal{R}$ then $R = \overline{\operatorname{int} R}$;
2. if $x, y \in R \in \mathcal{R}$ then $[x, y] \in R$;
3. if $x \in \operatorname{int} R_i \cap f^{-1} \operatorname{int} R_j$ then

$$f(W^s_\varepsilon(x) \cap R_i) \subset W^s_\varepsilon(fx) \cap R_j \quad \text{and} \quad W^u_\varepsilon(fx) \cap R_j \subset f(W^u_\varepsilon(x) \cap R_i). \tag{20}$$

For each $R \in \mathcal{R}$, we denote by R_* the set of points in R which return infinitely many times to R. By the Poincaré recurrence theorem we have $\mu(R_*) = \mu(R)$. For each $x \in R_*$ we set

$$T_R(x) \stackrel{\text{def}}{=} \inf\{k \in \mathbb{N} : f^k x \in R\} < \infty.$$

We define the *induced map* (or *first return map*) $f_R \colon R_* \to R_*$ on R_* by

$$f_R x = f^{T_R(x)} x. \tag{21}$$

Given $z \in \operatorname{int} R$ we define the *unstable map* $f_{z,u}$ on $W^u_{R_*}(z) \stackrel{\text{def}}{=} W^u_\varepsilon(z) \cap R_*$ (see Figure 2) by

$$\begin{aligned} f_{z,u} \colon W^u_{R_*}(z) &\to W^u_{R_*}(z) \\ x &\mapsto [f_R x, x]. \end{aligned}$$

Observe that $f_{z,u} x = [f_R x, z]$.

We denote by $\tau_r^u(x,\mathcal{R})$ the return time into the ball $B^u(x,r)$ with respect to the map $f_{z,u}$, i.e.

$$\tau_r^u(x,\mathcal{R}) = \inf\{k \in \mathbb{N} : {f_{z,u}}^k x \in B^u(x,r)\}, \tag{22}$$

and define the corresponding lower and upper recurrence rates by

$$\underline{R}^u(x,\mathcal{R}) = \varliminf_{r\to 0} \frac{\log \tau_r^u(x,\mathcal{R})}{-\log r} \quad \text{and} \quad \overline{R}^u(x,\mathcal{R}) = \varlimsup_{r\to 0} \frac{\log \tau_r^u(x,\mathcal{R})}{-\log r}.$$

Let also d_μ^u be as in Section 3.2. The following theorem relates the recurrence rate of the unstable map to the unstable dimension of the measure μ, and is crucial to the proof of Theorem 9.

Theorem 10. *Let Λ be a compact locally maximal hyperbolic set of a topologically mixing $C^{1+\alpha}$ diffeomorphism, and μ an equilibrium measure of a Hölder continuous function. Given a Markov partition $\mathcal{R}$ of Λ, and $R \in \mathcal{R}$, for μ-almost every $z \in R$ and μ_z^u-almost every $x \in W_R^u(z)$ we have*

$$\underline{R}^u(x,\mathcal{R}) = \overline{R}^u(x,\mathcal{R}) = \dim_H \mu_z^u = \underline{\dim}_B \mu_z^u = \overline{\dim}_B \mu_z^u = d_\mu^u.$$

Proof. Define inductively the return times of the f-orbit of x to the set R by $T_R^0(x) = 0$ and

$$T_R^n(x) = T_R^{n-1}(f_R x) + T_R(x) \tag{23}$$

for each $n \in \mathbb{N}$. For each $p \in \mathbb{N}$ we define a new partition of Λ by $\mathcal{R}^p = \bigvee_{k=0}^{p-1} f^{-k}\mathcal{R}$. We also define a partition $\mathcal{Z}$ of the set $W_{R_*}^u(z)$ by

$$\{Z \cap W_{R_*}^u(z) : Z \subset R \text{ and exists } p \geq 1 \text{ such that } Z \in \mathcal{R}^p \text{ and } T_R|_Z = p\}. \tag{24}$$

For each $n \in \mathbb{N}$ we consider the new partition $\mathcal{Z}_n \overset{\text{def}}{=} \bigvee_{k=0}^{n-1} {f_{z,u}}^{-k}\mathcal{Z}$. It follows immediately from the construction that $T_R(y) = T_R(x)$ whenever $y \in \mathcal{Z}(x)$. Therefore, for each $n \in \mathbb{N}$ we also have $T_R^n(y) = T_R^n(x)$ whenever $y \in \mathcal{Z}_n(x)$.

The following lemma provides crucial information about the geometric structure of the unstable map.

Lemma 1. *The following properties hold:*

1. *$\mathcal{Z}$ is a countable Markov partition of $W_{R_*}^u(z)$, with respect to the map $f_{z,u} \colon W_{R_*}^u(z) \to W_{R_*}^u(z)$, such that $f_{z,u}|Z$ is onto for each $Z \in \mathcal{Z}$;*
2. *there exists $\lambda > 0$ such that for each sufficiently large $n \in \mathbb{N}$ we have*

$$\sup_{Z\in\mathcal{Z}_n} \operatorname{diam}_{d^u} Z < \mathrm{e}^{-\lambda n};$$

3. *there exist $\theta > 0$ and $\alpha \in (0,1]$ such that if $n \in \mathbb{N}$, $Z \in \mathcal{Z}_n$, and $x, y \in Z$ then*

$$d^u({f_{z,u}}^n x, {f_{z,u}}^n y) \leq \exp(\theta T_R^n(x)) d^u(x,y)^\alpha.$$

Proof of Lemma 1. The first statement follows easily from the definitions. For the second statement observe first that each element Z of $\mathcal{Z}_n$ is contained in some element of the partition $\mathcal{R}^{T_R^n(x)}$ where $x \in \mathcal{Z}_n$. Choose $\lambda > 0$ such that

$$\varlimsup_{n\to\infty} \frac{1}{n} \log d^u(f^{-n}x, f^{-n}y) < -2\lambda$$

for all sufficiently close x and y on the same unstable manifold. Since $T_R^n \geq n$ we obtain

$$\sup_{Z\in\mathcal{Z}_n} \operatorname{diam}_{d^u} Z < \sup_{Z\in\mathcal{Z}_n} \mathrm{e}^{-\lambda T_R^n(x)} \leq \mathrm{e}^{-\lambda n}$$

whenever $Z \in \mathcal{Z}_n$ and $x \in Z$, for all sufficiently large n. It follows immediately from the construction that $f_{z,u}|_Z$ is onto for each $Z \in \mathcal{Z}$.

We now establish the third statement in the lemma. The Markov property of the partition $\mathcal{R}$ implies (through the first inclusion in (20)) that

$$f_{z,u}{}^n x = \underbrace{[f_R[f_R \cdots [f_R x, x], \ldots, x], x]}_{n \text{ times}} = [f_R{}^n x, x] = [f^{T_R^n(x)}x, x]. \tag{25}$$

Choose $\kappa > 0$ such that e^κ is a Lipschitz constant for f. Let $Z \in \mathcal{Z}_n$ and $x,\, y \in Z$. Since $T_R^n(x) = T_R^n(y)$ we obtain

$$d(f_R{}^n x, f_R{}^n y) = d(f^{T_R^n(x)}x, f^{T_R^n(x)}y) \leq \exp(\kappa T_R^n(x)) d(x, y).$$

Since the product structure $[\cdot,\cdot]$ is Hölder continuous, there exist constants $c > 1$ and $\alpha \in (0, 1]$ such that

$$\begin{aligned} d^u(f_{z,u}{}^n x, f_{z,u}{}^n y) &= d^u([f_R{}^n x, z], [f_R{}^n y, z]) \\ &\leq c d(f_R{}^n x, f_R{}^n y)^\alpha \\ &\leq c(\exp(\kappa T_R^n(x)))^\alpha d(x, y)^\alpha \\ &\leq c \exp(\kappa\alpha T_R^n(x)) d^u(x, y)^\alpha. \end{aligned}$$

Taking $\theta = \kappa\alpha + \log c$ we obtain the third statement. This completes the proof of the lemma. □

Notice that the second property in Lemma 1 corresponds to hypothesis 2 in Theorem 5.

For each $n \in \mathbb{N}$ we define still a new partition $\mathcal{R}_n = \bigvee_{p=0}^n f^p\mathcal{R}$. The partition

$$\mathcal{R}_\infty = \lim_{n\to\infty} \mathcal{R}_n = \{W_R^u(z) \colon z \in R \in \mathcal{R}\}$$

is composed of local unstable manifolds, and gives rise to a family of conditional measures μ_z^u for μ-almost every $z \in \Lambda$. The measure μ_z^u has the explicit representation

$$\mu_z^u(A) = \lim_{p\to\infty} \frac{\mu(A \cap \mathcal{R}_p(z))}{\mu(\mathcal{R}_p(z))}, \tag{26}$$

for μ-almost every $z \in \Lambda$ and every measurable set $A \subset M$.

Each measure μ_z^u can be seen as a measure on $W_R^u(z) = \mathcal{R}_\infty(z)$. However, this measure may not be invariant under the unstable map $f_{z,u}$ on $W_{R_*}^u(z)$. We shall construct another measure m_z^u equivalent to $\mu_{z,u}$ which is $f_{z,u}$-invariant.

Given a set $A \subset W_R^u(z)$ we write

$$[A, R] \stackrel{\text{def}}{=} \{[a, y] \colon a \in A, y \in R\}.$$

We define a new measure m_z^u on $W_R^u(z)$ by

$$m_z^u(A) \stackrel{\text{def}}{=} \frac{\mu([A, R])}{\mu(R)}, \tag{27}$$

for each measurable set $A \subset W^u_R(z)$. Since $\mu(\partial\mathcal{R}) = 0$ for any equilibrium measure μ (see Appendix A), the new measure m^u_z is a well-defined probability measure on $W^u_R(z)$ such that $m^u_z(\partial\mathcal{Z}) = 0$ for μ-almost every $z \in \Lambda$. Here the boundary $\partial\mathcal{Z}$ is computed with respect to the induced topology on $W^u_R(z)$.

Lemma 2. *There exists a constant $\Gamma > 0$ such that for μ-almost every $z \in R$ the following properties hold:*

1. *m^u_z is an ergodic $f_{z,u}$-invariant measure on $W^u_R(z)$;*
2. *the measures m^u_z and μ^u_z are equivalent, and*

$$\Gamma^{-1} < \frac{d\mu^u_z}{dm^u_z} < \Gamma.$$

Proof of Lemma 2. Let $z \in \operatorname{int} R$. Since

$$[{f_{z,u}}^{-1}A, R] = {f_R}^{-1}[A, R],$$

the invariance of m^u_z follows immediately from the f_R-invariance of the measure $\mu|_R$. The ergodicity of m^u_z follows from the ergodicity of $\mu|_R$.

We now establish the second property. Since the Markov partition is a generating partition, it is enough to verify the equivalence of measures on cylinders, i.e. on elements of the partitions $\mathcal{R}_m \vee \mathcal{R}^n$ for each n and m. Observe that for each $x \in W^u_R(z)$ we have $\mathcal{R}_p(x) = \mathcal{R}_p(z)$ for any $p \in \mathbb{N}$. Let m and n be positive integers, and consider the cylinder $Z = \mathcal{R}_m(x) \cap \mathcal{R}^n(x)$. By the Gibbs property of μ there exists a constant $a > 0$ (independent of m, n, and x) such that

$$a^{-1}\mu(\mathcal{R}_m(x))\mu(\mathcal{R}^n(x)) \le \mu(Z) \le a\mu(\mathcal{R}_m(x))\mu(\mathcal{R}^n(x)).$$

Dividing by $\mu(\mathcal{R}_m(x))$ and letting $m \to \infty$, it follows from (26) that

$$a^{-1}\mu(\mathcal{R}^n(x)) \le \mu^u_z(Z) \le a\mu(\mathcal{R}^n(x))$$

for every $n \in \mathbb{N}$. Since $[Z, R] = \mathcal{R}^n(x)$, it follows from (27) that $m^u_z(Z) = \mu(\mathcal{R}^n(x))/\mu(R)$, and hence

$$a^{-1}\mu(R) \le \frac{\mu^u_z(Z)}{m^u_z(Z)} \le a\mu(R)$$

Setting $\Gamma = a\mu(R)^{-1}$ we obtain the desired statement. □

We now establish hypothesis 3 in Theorem 5.

Lemma 3. *For μ-almost every $z \in \Lambda$ and m^u_z-almost every $x \in W^u_R(z)$ there exists $\gamma > 0$ such that if $n \in \mathbb{N}$ is sufficiently large then*

$$B(x, \mathrm{e}^{-\gamma n}) \subset \mathcal{Z}_n(x). \tag{28}$$

Proof of Lemma 3. Set $\mu_R = \mu|_R/\mu(R)$. By Kac's lemma the induced dynamical system (f_R, μ_R) is ergodic and

$$\int_R T_R(x)\, d\mu_R(x) = \frac{1}{\mu(R)}. \tag{29}$$

Observe that

$$T^n_R(x) = \sum_{k=0}^{n-1} T_R({f_R}^k x) \tag{30}$$

for each $n \in \mathbb{N}$ and $x \in R$. By Birkhoff's ergodic theorem we obtain

$$\mu\left(\left\{x \in R : \lim_{n\to\infty} \frac{1}{n} T_R^n(x) = \frac{1}{\mu(R)}\right\}\right) = \mu(R),$$

and thus (see (26))

$$\mu_z^u\left(\left\{x \in W_R^u(z) : \lim_{n\to\infty} \frac{1}{n} T_R^n(x) = \frac{1}{\mu(R)}\right\}\right) = 1$$

for μ-almost every $z \in \Lambda$. By Lemma 2 we conclude that

$$m_z^u\left(\left\{x \in W_R^u(z) : \lim_{n\to\infty} \frac{1}{n} T_R^n(x) = \frac{1}{\mu(R)}\right\}\right) = 1$$

for μ-almost every z. Hence, for m_z^u-almost every $x \in W_R^u(z)$ there exists $\delta_x > 1/\mu(R)$ such that $T_R^n(x) < \delta_x n$ for all $n \in \mathbb{N}$. Furthermore δ_x can be chosen in such a way that $x \mapsto \delta_x$ is measurable.

Let $\varepsilon > 0$, $\delta > 0$, and

$$Y_0 \stackrel{\text{def}}{=} \{x \in W_R^u(z)\colon \delta_x < \delta\}.$$

We have $m_z^u(Y_0) > 1 - \varepsilon$ for all sufficiently large δ. Denote by $\mathcal{P}_n \subset \mathcal{Z}$ the collection of elements $Z \in \mathcal{Z}$ such that $T_R|_{[Z,R]} \le n$. If $x \in Y_0$ then

$$\mathcal{Z}({f_{z,u}}^{n-1}x) \in \mathcal{P}_{\delta n} \tag{31}$$

for any $n \in \mathbb{N}$ (since $T_R({f_{z,u}}^{n-1}x) \le T_R^n(x) < \delta_x n < \delta n$), where $\mathcal{Z}(x)$ denotes the element of $\mathcal{Z}$ containing x. Furthermore, by the construction of $\mathcal{Z}$ in (24) we have

$$[\mathcal{P}_n, R] \subset \mathcal{R}^1 \cup \cdots \cup \mathcal{R}^n.$$

We now want to show that the orbit of a typical point in Y_0 stays far away from $\partial\mathcal{Z}$. Using the Markov property of the partition $\mathcal{R}$, we conclude that

$$[\partial\mathcal{P}_m, R] \subset \partial\mathcal{R}^m$$

for any integer $m \in \mathbb{N}$. The Hölder regularity of the product structure implies that there exists $\alpha > 0$ such that

$$[\{x \in W_R^u(z)\colon d^u(x, \partial\mathcal{P}_m) < r\}, R] \subset \{y \in R : d(y, \partial\mathcal{R}^m) < r^\alpha\}. \tag{32}$$

Let

$$\beta_0 = (1 + \log\max\{\|d_x f\| : x \in \Lambda\})\delta/\alpha,$$

and define

$$B_n = \left\{x \in Y_0 : d^u({f_{z,u}}^{n-1}x, \partial\mathcal{Z}({f_{z,u}}^{n-1}x)) \le \mathrm{e}^{-\beta_0 n}\right\}.$$

Using (31) we get

$$m_z^u(B_n) \le m_z^u\left(\left\{x \in Y_0 : d^u({f_{z,u}}^{n-1}x, \partial\mathcal{P}_{\delta n}) \le \mathrm{e}^{-\beta_0 n}\right\}\right),$$

and by the $f_{z,u}$-invariance of m_z^u we obtain

$$m_z^u(B_n) \le m_z^u\left(\left\{x \in W_R^u(z) : d^u(x, \partial\mathcal{P}_{\delta n}) \le \mathrm{e}^{-\beta_0 n}\right\}\right).$$

By (27), (32), and Proposition 16 in Appendix A, there exist constants $c > 0$ and $\nu > 0$ such that

$$m_z^u(B_n) \le \frac{1}{\mu(R)}\mu\left(\left\{x \in R : d(x, \partial\mathcal{R}^{\delta n}) \le \mathrm{e}^{-\alpha\beta_0 n}\right\}\right)$$
$$\le c\mathrm{e}^{-n\alpha\beta_0\nu/\delta} \le c\mathrm{e}^{-\nu n}$$

for every $n \in \mathbb{N}$. This implies that $\sum_{m\in\mathbb{N}} m_z^u(B_m) < \infty$. By the Borel–Cantelli lemma, for m_z^u-almost every $x \in Y_0$ we have that $x \notin B_m$ for all sufficiently large m, that is,

$$d^u({f_{z,u}}^{m-1}x, \partial\mathcal{Z}({f_{z,u}}^{m-1}x)) > \mathrm{e}^{-\beta_0 m}$$

for all sufficiently large m. Consequently, for some $\beta > \beta_0$ there exists a set $Y \subset Y_0$ of measure $m_z^u(Y) > 1 - 2\varepsilon$ such that

$$d^u({f_{z,u}}^{m-1}x, \partial\mathcal{Z}({f_{z,u}}^{m-1}x)) > \mathrm{e}^{-\beta m} \tag{33}$$

for all $m \in \mathbb{N}$ and each $x \in Y$ (recall that the boundary $\partial\mathcal{Z}$ has zero measure; see Appendix A).

Fix $\gamma > (\beta + \theta\delta)/\alpha$, with θ as in Lemma 1. Let $x \in Y$, $n \ge 2$, and $y \in B(x, \mathrm{e}^{-\gamma n})$. It is easy to verify that

$$\mathrm{e}^{k\theta\delta}\mathrm{e}^{-\alpha\gamma n} \le \mathrm{e}^{-\beta n} \tag{34}$$

for every $k \le n$. By (33) and (34) with $m = 1$ and $k = 0$, we have

$$d^u(x, \partial\mathcal{Z}(x)) > \mathrm{e}^{-\beta} > \mathrm{e}^{-\alpha\gamma n} > d^u(x, y).$$

Thus $y \in \mathcal{Z}(x)$. By Lemma 1 we obtain

$$d^u(f_{z,u}x, f_{z,u}y) \le \mathrm{e}^{\theta\delta} d(x, y)^\alpha \le \mathrm{e}^{\theta\delta}\mathrm{e}^{-\alpha\gamma n} \le \mathrm{e}^{-\beta n},$$

using (34) with $k = 1$. By (33) with $m = 2$ we have

$$d^u(f_{z,u}x, \partial\mathcal{Z}(f_{z,u}x)) > \mathrm{e}^{-2\beta} \ge d^u(f_{z,u}x, f_{z,u}y),$$

and hence $f_{z,u}y \in \mathcal{Z}(f_{z,u}x)$. Thus $y \in \mathcal{Z}_2(x)$. Again by Lemma 1 we obtain

$$d^u({f_{z,u}}^2x, {f_{z,u}}^2y) \le \mathrm{e}^{2\theta\delta}\mathrm{e}^{-\alpha\gamma n} \le \mathrm{e}^{-\beta n},$$

using (34) with $k = 2$. We can repeat successively the above argument to conclude that for every $m \le n$ we have

$$d^u({f_{z,u}}^{m-1}x, \partial\mathcal{Z}({f_{z,u}}^{m-1}x)) > \mathrm{e}^{-m\beta} \ge d^u({f_{z,u}}^m x, {f_{z,u}}^m y),$$

and hence ${f_{z,u}}^{m-1}x \in \mathcal{Z}({f_{z,u}}^{m-1}x)$. Thus $y \in \mathcal{Z}_m(x)$. This shows that (28) holds for any $x \in Y$ and $n \ge 2$. Since $m_z^u(Y) > 1 - 2\varepsilon$, the arbitrariness of $\varepsilon > 0$ implies the desired statement. □

We continue with the proof of the theorem. By Lemma 2, the measure m_z^u is an $f_{z,u}$-invariant ergodic measure. Observe that $[\mathcal{Z}, R]$ is a partition of R. Recall that for each $Z \in \mathcal{Z}$ there exists $p > 0$ such that $[Z, R] \in \mathcal{R}^p$ and $T_R|_{[Z,R]} = p$. The Gibbs property of the measure μ implies that there exists a constant $b>0$ such that $\mu([Z,R])>\mathrm{e}^{-bp}$, and hence $m_z^u(Z)>\mathrm{e}^{-bp}$ for every $Z\in\mathcal{Z}$. By (27), this implies that

$$\begin{aligned}-\sum_{Z\in\mathcal{Z}} m_z^u(Z)\log m_z^u(Z) &= \sum_{p>0}\sum_{Z\in\mathcal{Z}:T_R|_{[Z,R]}=p} m_z^u(Z)(-\log m_z^u(Z))\\ &\le \sum_{p>0}\frac{bp}{\mu(R)}\sum_{Z\in\mathcal{Z}:T_R|_{[Z,R]}=p}\mu([Z,R])\\ &= \frac{b}{\mu(R)}\sum_{p>0}p\mu\left(\bigcup_{Z\in\mathcal{Z}:T_R|_{[Z,R]}=p}[Z,R]\right)\\ &= \frac{b}{\mu(R)}\int_R T_R\,d\mu.\end{aligned}$$

It follows from Kac's lemma (see (29)) that $H_{m_z^u}(\mathcal{Z})\le b/\mu(R)<\infty$.

We now verify that the system $(f_{z,u},m_z^u)$ satisfies the remaining hypotheses of Theorem 5:

1. Observe that

   $$[\mathcal{Z}_{n+m}(x),R]=[\mathcal{Z}_n(x),R]\cap f^{-p}[\mathcal{Z}_m(y),R],$$

   with $[\mathcal{Z}_n(x),R]\in\mathcal{R}^p$ and $y={f_{z,u}}^n x=f^p x$. It follows from (27) and the Gibbs property of μ that there there exists a constant $\kappa>0$ (independent of m, n, and x) such that

   $$m_z^u(\mathcal{Z}_{n+m}(x))\le\kappa m_z^u(\mathcal{Z}_n(x))m_z^u(\mathcal{Z}_m({f_{z,u}}^n x)).$$

   This shows that the hypothesis 1 in Theorem 5 holds.
2. Hypothesis 2 is the statement 2 of Lemma 1.
3. Hypothesis 3 is the content of Lemma 3.

We can now apply Theorem 5 to conclude that the identities in (12) hold, i.e.

$$\underline{R}^u(x,\mathcal{R})=\underline{d}_{m_z^u}(x)\quad\text{and}\quad\overline{R}^u(x,\mathcal{R})=\overline{d}_{m_z^u}(x)\tag{35}$$

for μ-almost every $z\in R$ and m_z^u-almost every $x\in W_R^u(z)$.

In [6], Ledrappier and Young showed that there exists a constant d_μ^u (see also Section 3.2) such that

$$\underline{d}_{\mu_x^u}(x)=\overline{d}_{\mu_x^u}(x)=d_\mu^u\tag{36}$$

for μ-almost every $x\in\Lambda$. Recall that for each $x\in W_R^u(z)$ we have $\mathcal{R}_p(x)=\mathcal{R}_p(z)$ for any $p\in\mathbb{N}$. Hence, it follows from (26) and (36) that for μ-almost every $z\in\Lambda$ and μ_z^u-almost every $x\in W_R^u(z)$, we have

$$\underline{d}_{\mu_z^u}(x)=\overline{d}_{\mu_z^u}(x)=d_\mu^u.\tag{37}$$

It follows now immediately from a criterion of Young in [10] that

$$\dim_H\mu_z^u=\underline{\dim}_B\mu_z^u=\overline{\dim}_B\mu_z^u=d_\mu^u.$$

By Lemma 2 the measures m_z^u and μ_z^u are equivalent for μ-almost every $z\in R$, and hence

$$\underline{d}_{m_z^u}(x)=\underline{d}_{\mu_z^u}(x)\quad\text{and}\quad\overline{d}_{m_z^u}(x)=\overline{d}_{\mu_z^u}(x)\tag{38}$$

for every $x \in W^u_R(z)$. Combining (35), (38), and (37) we conclude that

$$\underline{R}^u(x,\mathcal{R}) = \overline{R}^u(x,\mathcal{R}) = \underline{d}_{m^u_z}(x) = \overline{d}_{m^u_z}(x) = d^u_\mu$$

for μ-almost every $z \in R$ and μ^u_z-almost every $x \in W^u_R(z)$. This completes the proof of the theorem. □

Theorem 10 is a version of Theorem 4 for the induced dynamics on each rectangle of the Markov partition. With slight changes one can also establish a version of Theorem 10 for a certain "stable map" (which coincides with the unstable map with respect to f^{-1}).

5. Proofs of the results in Sections 1–3

5.1. **Proofs of the results in Section 3.**

Proof of Theorem 9. We shall use the successive return times of the f-orbit of x to the set R defined by $T^0_R(x) = 0$ and by (23) for each $n \in \mathbb{N}$. Let $\mathcal{R}$ be a Markov partition for (Λ, f). Given $R \in \mathcal{R}$ and $x \in R$ we consider the integer $\tau^u_r(x,\mathcal{R})$ defined by (22). It follows readily from (25) that

$$\tau^u_r(x,\mathcal{R}) = \inf\{k \in \mathbb{N} : d^u([{f_R}^k x, x], x) < r\},$$

with f_R as in (21). Therefore

$$T_R^{\tau^u_r(x,\mathcal{R})}(x) = \inf\{n \in \mathbb{N} : f^n x \in R \text{ and } d^u([f^n x, x], x) < r\}. \tag{39}$$

Furthermore, since μ is ergodic it follows from Kac's Lemma (see (29)) and (30) that

$$\lim_{n\to\infty} \frac{1}{n} T^n_{\mathcal{R}(x)}(x) = \frac{1}{\mu(\mathcal{R}(x))}$$

for μ-almost every $x \in \Lambda$, where $\mathcal{R}(x)$ denotes the element of $\mathcal{R}$ containing the point x. Therefore

$$\lim_{n\to\infty} \frac{\log T^n_{\mathcal{R}(x)}(x)}{\log n} = 1 \tag{40}$$

for μ-almost every $x \in \Lambda$.

Fix $\rho > 0$ and consider now two Markov partitions $\mathcal{R}_+$ and $\mathcal{R}_-$ for (Λ, f). We assume that $\mathcal{R}_-$ has diameter at most ρ (it is well known that there exist Markov partitions of arbitrarily small diameter), and define

$$\Lambda_\rho(\mathcal{R}_+) = \{x \in \Lambda \colon d(x, \partial\mathcal{R}_+) > \rho\}.$$

Observe that if $x \in \Lambda_\rho(\mathcal{R}_+)$ then

$$\mathcal{R}_-(x) \subset B(x,\rho) \cap \Lambda \subset \mathcal{R}_+(x), \tag{41}$$

where $\mathcal{R}_-(x)$ and $\mathcal{R}_+(x)$ denote respectively the elements of $\mathcal{R}_-$ and $\mathcal{R}_+$ containing x. Since Λ is an invariant set, the orbit of every point $x \in \Lambda$ is fully contained in Λ, and thus (even though the set $B(x,\rho)\cap\Lambda$ in (41) cannot in general be replaced by $B(x,\rho)$) one can use (41) and (39) to obtain

$$T_{\mathcal{R}_+(x)}^{\tau^u_r(x,\mathcal{R}_+)}(x) \le \tau^u_r(x,\rho) \le T_{\mathcal{R}_-(x)}^{\tau^u_r(x,\mathcal{R}_-)}(x)$$

for μ-almost every $x \in \Lambda_\rho(\mathcal{R}_+)$. It follows from (40) that

$$\underline{R}^u(x,\mathcal{R}_+) \le \underline{R}^u(x,\rho) \le \underline{R}^u(x,\mathcal{R}_-) \tag{42}$$

and

$$\overline{R}^u(x, \mathcal{R}_+) \le \overline{R}^u(x, \rho) \le \overline{R}^u(x, \mathcal{R}_-) \tag{43}$$

for μ-almost every $x \in \Lambda_\rho(\mathcal{R}_+)$. By Theorem 10 we have

$$\underline{R}^u(x, \mathcal{R}_-) = \overline{R}^u(x, \mathcal{R}_-) = d^u_\mu \quad \text{and} \quad \underline{R}^u(x, \mathcal{R}_+) = \overline{R}^u(x, \mathcal{R}_+) = d^u_\mu$$

for μ-almost every $x \in \Lambda$. We conclude from (42) and (43) that

$$\underline{R}^u(x, \rho) = \overline{R}^u(x, \rho) = d^u_\mu \tag{44}$$

for μ-almost every $x \in \Lambda_\rho(\mathcal{R}_+)$ and all sufficiently small $\rho > 0$. Since $\partial\mathcal{R}_+$ has zero measure (see Appendix A), the set $\bigcup_{\rho>0} \Lambda_\rho(\mathcal{R}_+)$ has full μ-measure, and hence the identities in (44) hold for μ-almost every $x \in \Lambda$ and all sufficiently small $\rho > 0$ (possibly depending on x).

One can obtain a version of Theorem 10 for the stable direction (by replacing f by f^{-1}, and the index u by s everywhere). Using similar arguments to those above, this can be used to show that

$$\underline{R}^s(x, \rho) = \overline{R}^s(x, \rho) = d^s_\mu \tag{45}$$

for μ-almost every $x \in \Lambda$ and all sufficiently small $\rho > 0$ (possibly depending on x) . It follows from (44) and (45) that the limits in (16) and (17) are not necessary provided that ρ is taken sufficiently small. Therefore, we have

$$R^s(x) = \underline{R}^s(x, \rho) = \overline{R}^s(x, \rho) = d^s_\mu$$

and

$$R^u(x) = \underline{R}^u(x, \rho) = \overline{R}^u(x, \rho) = d^u_\mu$$

for μ-almost every $x \in \Lambda$ and all sufficiently small $\rho > 0$ (possibly depending on x). This completes the proof of the theorem. □

Proof of Theorem 7. It was established by Barreira, Pesin, and Schmeling in [1] that

$$\underline{d}_\mu(x) = \overline{d}_\mu(x) = d^u_\mu + d^s_\mu$$

for μ-almost every $x \in \Lambda$. In [2], Barreira and Saussol showed that

$$\underline{R}(x) = \underline{d}_\mu(x) \quad \text{and} \quad \overline{R}(x) = \overline{d}_\mu(x)$$

for μ-almost every $x \in \Lambda$. This implies that $R(x)$ is well-defined for μ-almost every $x \in \Lambda$. Finally, it follows from Theorem 9 that

$$R(x) = R^s(x) + R^u(x)$$

for μ-almost every $x \in \Lambda$. This establishes the first statement in the theorem. The second statement is an immediate consequence of the first one. □

5.2. **Proofs of the results in Section 2.** Let T be a Borel-measurable transformation on the metric space X, and μ a T-invariant probability measure on X. The *return time* of the point $y \in B(x, r)$ into $B(x, r)$ is defined by

$$\tau_r(y, x) \stackrel{\text{def}}{=} \inf\{k > 0 : d(T^k y, x) < r\}.$$

For each $x \in X$ and each r, $\varepsilon > 0$, we consider the set

$$A_\varepsilon(x, r) = \{y \in B(x, r) : \tau_r(y, x) \le \mu(B(x, r))^{-1+\varepsilon}\}. \tag{46}$$

We say that the measure μ has *long return time* (with respect to T) if

$$\varliminf_{r\to 0} \frac{\log \mu(A_\varepsilon(x,r))}{\log \mu(B(x,r))} > 1$$

for μ-almost every $x \in X$ and every sufficiently small $\varepsilon > 0$. The class of measures with long return time includes for example equilibrium measures supported on locally maximal hyperbolic sets (see [2] for a detailed discussion and for other examples of measures with long return time). We have the following criterion.

Proposition 11 ([2]). *Let T be a Borel-measurable transformation on a set $X \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$, and μ a T-invariant probability measure on X. If μ has long return time, and $\underline{d}_\mu(x) > 0$ for μ-almost every $x \in X$, then the identities in* (12) *hold for μ-almost every $x \in X$.*

We say that a measure μ is *weakly diametrically regular* on a set $Z \subset X$ if there is a constant $\eta > 1$ such that for μ-almost every $x \in Z$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that if $r < \delta$ then

$$\mu(B(x, \eta r)) \le \mu(B(x,r)) r^{-\varepsilon}. \tag{47}$$

One can easily verify that if μ is a weakly diametrically regular measure on a set Z, then for *every* fixed constant $\eta > 1$, there exists $\delta = \delta(x, \varepsilon, \eta) > 0$ for μ-almost every $x \in Z$ and every $\varepsilon > 0$, such that (47) holds for every $r < \delta$. The following is a criterion for weak diametric regularity.

Proposition 12 ([2]). *Let μ be a probability measure on a measurable set $Z \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$. Then μ is weakly diametrically regular on Z.*

We define the *return time* of a set A into itself by

$$\tau(A) = \inf\{n \in \mathbb{N} : T^n A \cap A \ne \varnothing\}.$$

Saussol, Troubetzkoy, and Vaienti show in [8] that the return time of an element of the partition $\mathcal{Z}_n = \bigvee_{k=0}^{n-1} T^{-k}\mathcal{Z}$ into itself is "typically large", in the following sense.

Proposition 13 ([8]). *Let $T\colon W \to W$ be a measurable transformation preserving an ergodic probability measure μ. If $\mathcal{Z}$ is a finite or countable measurable partition with entropy $h_\mu(T, \mathcal{Z}) > 0$ then*

$$\varliminf_{n\to\infty} \frac{\tau(\mathcal{Z}_n(x))}{n} \ge 1$$

for μ-almost every $x \in W$.

We shall use these statements in the proof of Theorem 5.

Proof of Theorem 5. We first show that the entropy $h_\mu(g)$ is finite and nonzero. Set $\sigma_n = \sup\{\mu(Z)\colon Z \in \mathcal{Z}_n\}$. It follows from (11) and the non-atomicity of the measure μ that $\sigma_n \to 0$ as $n \to \infty$. Otherwise there would exist $x \in W$ and $\varepsilon > 0$ such that $\mu(\mathcal{Z}_n(x)) \to \varepsilon$ as $n \to \infty$. But using (11) we have $\bigcap_{n\in\mathbb{N}} \mathcal{Z}_n(x) = \{x\}$ and thus $\mu(\{x\}) = \varepsilon > 0$, which contradicts the non-atomicity of the measure μ. In particular, there exists $p \in \mathbb{N}$ such that $\sigma_p < 1/\kappa$. Using (10) we find that

$$\mu(\mathcal{Z}_{pn}(x)) \le (\kappa\sigma_p)^n,$$

for every $x \in W$ and every $n \in \mathbb{N}$. The Shannon–McMillan–Breiman theorem implies that

$$h_\mu(g) \geq \varliminf_{n\to\infty} \frac{\log \mu(\mathcal{Z}_{pn}(x))}{-pn} \geq -\frac{1}{p}\log(\kappa\sigma_p) > 0.$$

Furthermore, it follows from (11) that $\mathcal{Z}$ is a generating partition and hence $h_\mu(g) = h_\mu(g, \mathcal{Z}) \leq H_\mu(\mathcal{Z}) < \infty$.

Our assumptions insure that for μ-almost every point $x \in W$ there exists $\gamma > 0$ such that

$$B(x, \mathrm{e}^{-\gamma n}) \subset \mathcal{Z}_n(x) \subset B(x, \mathrm{e}^{-\lambda n})$$

for all sufficiently large n. The Shannon–McMillan–Breiman theorem implies that

$$\gamma \underline{d}_\mu(x) \geq h_\mu(g) \geq \lambda \overline{d}_\mu(x)$$

for μ-almost every $x \in W$. Since $0 < h_\mu(g) < \infty$ we conclude that $0 < \underline{d}_\mu(x) \leq \overline{d}_\mu(x) < \infty$ for μ-almost every $x \in W$.

Since $h_\mu(g, \mathcal{Z}) > 0$ we can apply Proposition 13. By hypothesis 3 in the theorem we conclude that for μ-almost every $x \in W$ there exists $\gamma > 0$ such that

$$\varliminf_{r\to 0} \frac{\tau(B(x,r))}{-\log r} = \varliminf_{n\to\infty} \frac{\tau(B(x,\mathrm{e}^{-\gamma n}))}{\gamma n} \geq \varliminf_{n\to\infty} \frac{\tau(\mathcal{Z}_n(x))}{\gamma n} \geq \frac{1}{\gamma}. \tag{48}$$

The first identity follows easily from the fact that given $r > 0$ there exists $n = n(r) \in \mathbb{N}$ such that $\mathrm{e}^{-\gamma(n+1)} < r \leq \mathrm{e}^{-\gamma n}$ and thus

$$\frac{\tau(B(x,\mathrm{e}^{-\gamma(n+1)}))}{\gamma n} > \frac{\tau(B(x,r))}{-\log r} > \frac{\tau(B(x,\mathrm{e}^{-\gamma n}))}{\gamma(n+1)}.$$

It follows from (48) that

$$B(x,r) \cap g^{-k}B(x,r) = \varnothing$$

whenever k is a positive integer such that $k < -\frac{1}{2\gamma}\log r$ and r is sufficiently small.

Set $B_k = \bigcup_{y\in B(x,r)} \mathcal{Z}_k(y)$ and write B_k as a disjoint union $\bigcup_{j=1}^N \mathcal{Z}_k(y_j)$. Choose also sets $Z_1, Z_2, \ldots \in \bigcup_{n\in\mathbb{N}} \mathcal{Z}_n$ such that $Z_i \cap Z_j = \varnothing \pmod 0$ for every $i \neq j$, and

$$B(x,r) = \bigcup_{\ell\in\mathbb{N}} Z_\ell \pmod 0.$$

The inequality (10) implies that

$$\begin{aligned} \mu(B_k \cap g^{-k}B(x,r)) &= \sum_{j=1}^N \sum_{\ell\in\mathbb{N}} \mu(\mathcal{Z}_k(y_j) \cap g^{-k}Z_\ell) \\ &\leq \sum_{j=1}^N \sum_{\ell\in\mathbb{N}} \kappa\mu(\mathcal{Z}_k(y_j))\mu(Z_\ell) \\ &= \kappa\mu(B_k)\mu(B(x,r)). \end{aligned}$$

By hypothesis 2 we have $B_k \subset B(x, r + \mathrm{e}^{-\lambda k})$ for all sufficiently small $r > 0$, and hence

$$\frac{\mu(B(x,r) \cap g^{-k}B(x,r))}{\mu(B(x,r))} \leq \kappa\mu(B(x, r + \mathrm{e}^{-\lambda k})). \tag{49}$$

By Proposition 12, if $k \geq -\frac{1}{\lambda}\log r$ then

$$\mu(B(x, r + \mathrm{e}^{-\lambda k})) \leq \mu(B(x, 2r)) \leq \mu(B(x, r))r^{-\varepsilon(r)},$$

where $\varepsilon(r) \to 0$ as $r \to 0$. Note that $\varepsilon(r)$ may depend on x. By rechoosing $\varepsilon(r)$ if necessary we can assume that if $k \geq -\frac{1}{2\gamma}\log r$ then (since $\lambda/\gamma \leq 1$)

$$\mu(B(x, r + \mathrm{e}^{-\lambda k})) \leq \mu(B(x, r^{\frac{\lambda}{3\gamma}})) \leq r^{\frac{\lambda \underline{d}_\mu(x)}{3\gamma}} r^{-\varepsilon(r)}.$$

Combining these estimates with (49), and possibly rechoosing $\varepsilon(r)$, we obtain (see (46))

$$\begin{aligned}
\frac{\mu(A_\varepsilon(x,r))}{\mu(B(x,r))} &\leq \sum_{k=-\frac{1}{2\gamma}\log r}^{-\frac{1}{\lambda}\log r} r^{\frac{\lambda \underline{d}_\mu(x)}{3\gamma}} r^{-\varepsilon(r)} + \sum_{k=-\frac{1}{\lambda}\log r}^{\mu(B(x,r))^{-1+\varepsilon}} \mu(B(x,r))r^{-\varepsilon(r)} \\
&\leq \left(-\frac{1}{\lambda} + \frac{1}{2\gamma}\right)\log r \left(\mu(B(x,r))^{\frac{1}{\overline{d}_\mu(x)+\varepsilon}}\right)^{\frac{\lambda \underline{d}_\mu(x)}{3\gamma}} r^{-\varepsilon(r)} \\
&\quad + \left(\mu(B(x,r))^{-1+\varepsilon} + \frac{1}{\lambda}\log r\right)\mu(B(x,r))r^{-\varepsilon(r)} \\
&\leq \left[\mu(B(x,r))^{\frac{\lambda \underline{d}_\mu(x)}{3\gamma(\overline{d}_\mu(x)+\varepsilon)}} + \mu(B(x,r))^{\varepsilon}\right] r^{-2\varepsilon(r)}
\end{aligned}$$

for all sufficiently small $r > 0$. Since $\underline{d}_\mu(x) > 0$ for μ-almost every $x \in W$, this readily implies that

$$\varliminf_{r\to 0} \frac{\log \mu(A_\varepsilon(x,r))}{\log \mu(B(x,r))} \geq 1 + \min\left\{\frac{\lambda \underline{d}_\mu(x)}{3\gamma(\overline{d}_\mu(x)+\varepsilon)}, \varepsilon\right\} > 1$$

for μ-almost every $x \in W$, and thus the measure μ has long return time. The desired statement follows now immediately from Proposition 11. □

Proof of Theorem 4. Let $\mathcal{Z}$ be a (finite) Markov partition of X (with respect to the map T). Clearly $H_\mu(\mathcal{Z}) < \infty$.

We want to verify the remaining hypotheses in Theorem 5. Under the assumptions in the theorem, each equilibrium measure of a Hölder continuous function possesses the Gibbs property, and thus there exists a constant $\kappa > 0$ such that

$$\kappa^{-1}\mu(\mathcal{Z}_n(x))\mu(\mathcal{Z}_m(T^n x)) \leq \mu(\mathcal{Z}_{n+m}(x)) \leq \kappa\mu(\mathcal{Z}_n(x))\mu(\mathcal{Z}_m(T^n x))$$

for every n, $m \in \mathbb{N}$ and $x \in X$. In particular hypothesis 1 in Theorem 5 holds with

$$\lambda < -\sup_{x\in X} \varlimsup_{n\to\infty} \frac{1}{n}\log\|(d_x T)^{-1}\|.$$

Since T is expanding the diameter of $\mathcal{Z}_n$ converges exponentially fast to zero, and hypothesis 2 holds. Hypothesis 3 follows from Proposition 16 in Appendix A and the discussion thereafter. The desired result follows now immediately from Theorem 5. □

5.3. **Proofs of the results in Section 1.**

Proof of Theorem 1. Each toral endomorphism preserves the Lebesgue measure λ in $\mathbb{T}^d$, and $\underline{d}_\mu(x) = \overline{d}_\mu(x) = d$ for every $x \in \mathbb{T}^d$. The statement is thus an immediate consequence of Theorem 4. □

Proof of Theorem 2. We want to apply Theorem 5. Consider the T-invariant ergodic probability measure μ defined in (13). Let $\mathcal{Z}$ be the partition into intervals $I_n = (\frac{1}{n+1}, \frac{1}{n})$ for each $n \in \mathbb{N}$. One can easily verify that $H_\mu(\mathcal{Z}) < \infty$ and that $\sup_{Z \in \mathcal{Z}_n} \operatorname{diam} Z$ tends exponentially fast to zero as $n \to \infty$.

It is well known that there exists $\Delta > 0$ such that

$$\Delta^{-1} |(T^n)'(y)| \le |(T^n)'(x)| \le \Delta |(T^n)'(y)|$$

for every $x \in (0,1)$, $n \in \mathbb{N}$, and $y \in \mathcal{Z}_n(x)$. Furthermore, we have $T^n \mathcal{Z}_n(x) = (0,1)$ for each $x \in (0,1)$ and each $n \in \mathbb{N}$. Using this information it is straightforward to show that there exists a constant $\Gamma > 0$ such that

$$\Gamma^{-1} \le \mu(\mathcal{Z}_n(x)) |(T^n)'(x)| \le \Gamma$$

for every $x \in (0,1)$ and $n \in \mathbb{N}$. This guarantees that the inequality (10) holds with $\kappa = \Gamma^3$.

It remains to verify hypothesis 3 in Theorem 5. Using the invariance of μ we obtain

$$\sum_{n>0} \mu(\{x \in (0,1) : d(T^n x, \{0,1\}) < \mathrm{e}^{-n}\}) =$$
$$= \sum_{n>0} \mu([0, \mathrm{e}^{-n}] \cup [1 - \mathrm{e}^{-n}, 1]) < \infty.$$

By the Borel–Cantelli lemma we conclude that for μ-almost every $x \in (0,1)$ we have $d(T^n x, \partial T^n \mathcal{Z}_n(x)) \ge \mathrm{e}^{-n}$ for all sufficiently large $n \in \mathbb{N}$. Therefore for μ-almost every $x \in (0,1)$ we have

$$d(x, \partial \mathcal{Z}_n(x)) \ge \mathrm{e}^{-n} (\Delta |(T^n)'(x)|)^{-1} \tag{50}$$

for all sufficiently large $n \in \mathbb{N}$. By Birkhoff's ergodic theorem we have

$$\lambda \stackrel{\text{def}}{=} \int_0^1 \log |T'| \, d\mu = \lim_{n \to \infty} \frac{1}{n} \log |(T^n)'(x)| < \infty$$

for μ-almost every $x \in (0,1)$. Choose $\gamma > 1 + \lambda$. By (50) we obtain $B(x, \mathrm{e}^{-\gamma n}) \subset \mathcal{Z}_n(x)$ for μ-almost every $x \in (0,1)$ and all sufficiently large $n \in \mathbb{N}$. We have thus verified all the hypothesis in Theorem 5.

Since the measure μ is equivalent to Lebesgue, we have $\underline{d}_\mu(x) = \overline{d}_\mu(x) = 1$ for every $x \in (0,1)$. The desired result follows now immediately from Theorem 5. □

Proof of Theorem 3. The desired statement is contained in Theorem 9. □

Appendix A. Neighborhood of Markov partitions

In this appendix we study the boundaries of Markov partitions for hyperbolic sets. Let $\mathcal{R}$ be such a partition. It is well known that $\partial \mathcal{R}$ has zero measure with respect to any equilibrium measure. This is a simple consequence of the fact that $\partial \mathcal{R}$ is a closed nowhere dense set.

On the other hand, in Section 5 above we require an estimate on the measure of the ε-neighborhood of $\partial\mathcal{R}$ (see the proofs of Theorem 10 and Lemma 3). More precisely, in Section 4 we started with a hyperbolic set and constructed with an induction procedure a system having countably many components. It is essential for us to estimate the measure of the neighborhoods of the singularities, which in fact is related to the measure of the boundary of the original Markov partition for the hyperbolic set.

While such an estimate may be easier to obtain when each element of $\mathcal{R}$ has a piecewise smooth boundary (such as in the case of hyperbolic toral automorphisms of the 2-torus), it is well known that Markov partitions may have a very complicated boundary. In particular, it was first discovered by Bowen in [4] that $\partial\mathcal{R}$ need not be piecewise smooth (Bowen considered hyperbolic toral automorphisms of the 3-torus).

Nevertheless, we are able to prove below that the measure of the ε-neighborhood of the boundary of a Markov partition (for repellers or locally maximal hyperbolic sets, and arbitrary Gibbs measures) is at most polynomial in ε. We also consider expansive homeomorphisms with the specification property. The proofs rely entirely on the thermodynamic formalism.

Let $f\colon X \to X$ be a homeomorphism of the compact metric space (X, d). For each $n \in \mathbb{N}$ we define a new distance d_n on X by

$$d_n(x, y) = \max\{d(f^k x, f^k y) : k = 0, \dots, n-1\}.$$

We denote by $B_n(x, \delta)$ the ball centered at x of radius δ with respect to the distance d_n. Given a function $\varphi\colon X \to \mathbb{R}$ and $n \in \mathbb{N}$ we define the new function

$$\varphi_n(x) = \varphi(x) + \cdots + \varphi(f^{n-1}x).$$

Let $V(X)$ be the family of continuous functions with bounded variation, i.e. the continuous functions $\varphi\colon X \to \mathbb{R}$ such that there exist $\varepsilon > 0$ and $\delta > 0$ such that for each $n \in \mathbb{N}$ we have

$$|\varphi_n(x) - \varphi_n(y)| \le \varepsilon \text{ whenever } d_n(x, y) \le \delta.$$

When f is an expansive homeomorphism with the specification property, each function $\varphi \in V(X)$ has a unique equilibrium measure that we shall denote by μ_φ. We denote by $\eta(f)$ the supremum of the expansivity constants for f. We shall denote the topological pressure of a function φ with respect to the dynamical system $f|X$ by $P_X(\varphi)$. Since we only need to consider compact f-invariant sets we shall use Ruelle's classical definition.

Theorem 14. *Let $f\colon X \to X$ be an expansive homeomorphism with the specification property of the compact metric space X. For each $\eta < 8\eta(f)$, each $\varphi \in V(X)$, and each compact f-invariant set $K \subset X$ with $K \ne X$ there exist $c > 0$ and $\nu > 0$ such that if $n \in \mathbb{N}$ then*

$$\mu_\varphi(\{x \in X : d_n(x, K) < \eta\}) \le c e^{-\nu n}. \tag{51}$$

Proof. We first show that $P_K(\varphi) < P_X(\varphi)$. Since K is a compact f-invariant set, $f|_K$ is expansive, and $\varphi|_K \in V(K)$, there exists an equilibrium measure μ_K of the function $\varphi|_K$ with respect to the dynamical system $f|_K$. Clearly

$\operatorname{supp}\mu_K \subset K$. By the specification property, we have $\operatorname{supp}\mu_\varphi = X$, and hence $\mu_K \neq \mu_\varphi$. By the uniqueness of the equilibrium measure μ_φ we obtain

$$P_K(\varphi) = h_{\mu_K}(f) + \int_X \varphi \, d\mu_K < P_X(\varphi).$$

Let $E_n \subset K$ be any set such that

$$K \subset \bigcup_{y \in E_n} B_n(y, \eta). \tag{52}$$

Set $K_n = \{x \in X : d_n(x, K) < \eta\}$. It is clear that

$$K_n \subset \bigcup_{y \in E_n} B_n(y, 2\eta).$$

It is well known (see for example [5, Lemma 20.3.4]) that

$$\mu(B_n(y, 2\eta)) \leq \zeta \exp[-nP_X(\varphi) + \varphi_n(y)]$$

for some constant $\zeta > 0$ independent of $n \in \mathbb{N}$ and $y \in X$, and thus

$$\mu(K_n) \leq \sum_{y \in E_n} \mu(B_n(y, 2\eta)) \leq \zeta \sum_{y \in E_n} \mathrm{e}^{-nP_X(\varphi) + \varphi_n(y)}.$$

Since $f|_K$ is expansive we have

$$P_K(\varphi) = \lim_{n \to \infty} \frac{1}{n} \log \inf_{E_n} \sum_{y \in E_n} \mathrm{e}^{\varphi_n(y)},$$

where the infimum is taken over all sets E_n for which (52) holds. Therefore

$$\overline{\lim_{n \to \infty}} \frac{1}{n} \log \mu(K_n) \leq P_K(\varphi) - P_X(\varphi) < 0.$$

This completes the proof of the theorem. □

It follows from the proof of the theorem that it is possible to take any constant $\nu > 0$ in (51) such that $\nu < P_X(\varphi) - P_K(\varphi)$.

Let Λ be a hyperbolic set of a diffeomorphism f, and consider a Markov partition $\mathcal{R} = \{R_1, \ldots, R_p\}$ of Λ. The boundary of $\mathcal{R}$ is the union of the stable boundary

$$\partial^s \mathcal{R} = \bigcup_{i=1}^p \{x \in \partial R_i : W^u_\varepsilon(x) \cap \operatorname{int} R_i \neq \varnothing\},$$

and the unstable boundary

$$\partial^u \mathcal{R} = \bigcup_{i=1}^p \{x \in \partial R_i : W^s_\varepsilon(x) \cap \operatorname{int} R_i \neq \varnothing\}.$$

Moreover,

$$f(\partial^s \mathcal{R}) \subset \partial^s \mathcal{R} \quad \text{and} \quad f^{-1}(\partial^u \mathcal{R}) \subset \partial^u \mathcal{R}.$$

The sets $\partial^s \mathcal{R}$, $\partial^u \mathcal{R}$, and $\partial \mathcal{R} = \partial^s \mathcal{R} \cup \partial^u \mathcal{R}$ are nowhere dense, and hence they have zero measure with respect to any ergodic invariant measure with full support.

The following statement shows that the ε-neighborhood of $\partial \mathcal{R}$ is at most polynomial in ε.

Theorem 15. *Let Λ be a compact locally maximal hyperbolic set of a topologically mixing $C^{1+\alpha}$ diffeomorphism f, and μ an equilibrium measure of a Hölder continuous function. For any Markov partition $\mathcal{R}$ of Λ there exist constants $c > 0$ and $\nu > 0$ such that if $\varepsilon > 0$ then*

$$\mu(\{x \in \Lambda \colon d(x, \partial\mathcal{R}) < \varepsilon\}) \leq c\varepsilon^{\nu}. \tag{53}$$

Proof. Observe that

$$d_n(x, y) \leq \max\{\|d_x f\| : x \in \Lambda\}^n d(x, y) \tag{54}$$

for every $x, y \in \Lambda$. Since φ is Hölder continuous, we have $\varphi \in V(X)$. For any compact set $K \subset \Lambda$ such that $f(K) \subset K \neq \Lambda$, it follows from Theorem 14 and (54) that there exist constants $c = c(f, K) > 0$ and $\nu = \nu(f, K) > 0$ such that

$$\mu(\{x \in \Lambda \colon d(x, K) < \varepsilon\}) \leq c(f, K)\varepsilon^{\nu(f,K)}. \tag{55}$$

Using (55) with $K = \partial^s\mathcal{R}$ we obtain

$$\mu(\{x \in \Lambda \colon d(x, \partial^s\mathcal{R}) < \varepsilon\}) \leq c(f, \partial^s\mathcal{R})\varepsilon^{\nu(f,\partial^s\mathcal{R})}.$$

Similarly, using (55) with $K = \partial^u\mathcal{R}$ with respect to f^{-1} (note that equilibrium measures of a function φ with respect to f and to f^{-1} coincide) we obtain

$$\mu(\{x \in \Lambda \colon d(x, \partial^u\mathcal{R}) < \varepsilon\}) \leq c(f^{-1}, \partial^u\mathcal{R})\varepsilon^{\nu(f^{-1},\partial^u\mathcal{R})}.$$

Furthermore, we have

$$\{x \in \Lambda \colon d(x, \partial\mathcal{R}) < \varepsilon\} \subset \{x \in \Lambda \colon d(x, \partial^s\mathcal{R}) < \varepsilon\} \cup \{x \in \Lambda \colon d(x, \partial^u\mathcal{R}) < \varepsilon\}.$$

The desired statement follows immediately from this inclusion. □

Set $L = \max\{\|d_x f\| : x \in \Lambda\}$. It follows from the proof of Theorem 15 that it is possible to take any constant $\nu > 0$ in (53) such that

$$\nu < \frac{P_\Lambda(\varphi) - P_{I(\partial\mathcal{R})}(\varphi)}{\log L} \tag{56}$$

for some potential φ of μ, where

$$I(\partial\mathcal{R}) = \bigcup_{n\in\mathbb{Z}} f^n(\partial\mathcal{R})$$

is the invariant hull of $\partial\mathcal{R}$. When μ is the measure of maximal entropy one can set $\varphi = 0$, and thus the inequality in (56) becomes

$$\nu < \frac{h(f|\Lambda) - h(f|I(\partial\mathcal{R}))}{\log L} = \frac{h(f|\Lambda) - h(f|\partial\mathcal{R})}{\log L}.$$

Since $\partial\mathcal{R}$ is not an invariant set, the entropy $h(f|\partial\mathcal{R})$ must be computed using Pesin's definition of topological entropy as a Carathéodory dimension characteristic. Since $\partial\mathcal{R}$ is nowhere dense it always avoids some cylinder, and thus $h(f|\partial\mathcal{R}) < h(f|\Lambda)$.

We now consider the neighborhood of the iterated Markov partition $\mathcal{R}^n = \bigvee_{k=0}^{n-1} f^{-k}\mathcal{R}$.

Proposition 16. *Under the assumptions of Theorem 15, if $\sigma > L$ then*

$$\mu(\{x \in \Lambda \colon d(x, \partial\mathcal{R}^n) < 1/\sigma^n\}) \leq c(L/\sigma)^{\nu n}.$$

Proof. Since L is a Lipschitz constant for f, if $d(x,\partial\mathcal{R}^n)<1/\sigma^n$ then

$$d(f^k x,\partial\mathcal{R})<L^k/\sigma^n$$

for some $k<n$. By Theorem 15 we conclude that

$$\begin{aligned}\mu(\{x\in\Lambda\colon d(x,\partial\mathcal{R}^n)<1/\sigma^n\}) &\le \sum_{k<n}\mu(\{x\in\Lambda\colon d(f^k x,\partial\mathcal{R})<L^k/\sigma^n\})\\ &\le \sum_{k<n} c(L^k/\sigma^n)^\nu\\ &\le (L/\sigma)^{\nu n}.\end{aligned}$$

This completes the proof. □

With a little more care we could have shown (and this is optimal) that if λ denotes the largest Lyapunov exponent (with respect to the measure μ), then σ can be taken arbitrarily close to e^λ.

With straightforward modifications one can also establish the statements in Theorem 15 and Proposition 16 (as well as those described below) in the case of repellers.

A direct application of Proposition 16 gives

$$\sum_{n>0}\mu(\{x\in\Lambda\colon d(x,\partial\mathcal{R}^n)<1/\sigma^n\})\le\frac{c(L/\sigma)^\nu}{1-(L/\sigma)^\nu}<\infty,$$

and the Borel–Cantelli lemma shows that for μ-almost every $x\in\Lambda$, we have $B(x,\sigma^{-n})\cap\Lambda\subset R_n(x)$ for all sufficiently large $n\in\mathbb{N}$. Here $R_n(x)$ is the element of $\mathcal{R}^n$ containing x. This shows that hypothesis 3 in Theorem 5 is always satisfied when $\mathcal{Z}=\mathcal{R}$ is a Markov partition of a hyperbolic set.

Proposition 16 allows us to construct explicit open neighborhoods of $I(\partial\mathcal{R})$ with arbitrarily small measure. For example, if $\mathcal{R}$ is a Markov partition of a repeller then the open set

$$\bigcup_{n\in\mathbb{N}}\{x\in\Lambda\colon d(x,\partial\mathcal{R}^n)<1/\sigma^n\}$$

contains $I(\partial\mathcal{R})$ and has μ-measure at most $c(L/\sigma)^\nu/[1-(L/\sigma)^\nu]$.

References

1. L. Barreira, Ya. Pesin and J. Schmeling, *Dimension and product structure of hyperbolic measures*, Ann. of Math. (2) **149** (1999), 755–783.
2. L. Barreira and B. Saussol, *Hausdorff dimension of measures via Poincaré recurrence*, Comm. Math. Phys., to appear.
3. M. Boshernitzan, *Quantitative recurrence results*, Invent. Math. **113** (1993), 617–631.
4. R. Bowen, *Markov partitions are not smooth*, Proc. Amer. Math. Soc. **71** (1978), 130–132.
5. A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of mathematics and its applications 54, Cambridge University Press, 1995.
6. F. Ledrappier and L.-S. Young, *The metric entropy of diffeomorphisms. Part II: Relations between entropy, exponents and dimension*, Ann. of Math. (2) **122** (1985), 540–574.
7. D. Ornstein and B. Weiss, *Entropy and data compression schemes*, IEEE Trans. Inform. Theory **39** (1993), 78–83.
8. B. Saussol, S. Troubetzkoy and S. Vaienti, *No small return*, in preparation.

9. J. Schmeling and S. Troubetzkoy, *Pointwise dimension for regular hyperbolic measures for endomorphisms*, in preparation.
10. L.-S. Young, *Dimension, entropy and Lyapunov exponents*, Ergodic Theory Dynam. Systems **2** (1982), 109–124.

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