Centre de Physique Théorique*, CNRS Luminy, Case 907
F-13288 Marseille – Cedex 9

Fractal and statistical characteristics of recurrence times. [‡]

Vincent Penné[1], Benoît Saussol[1], Sandro Vaienti[1].

Dedicated to Giovanni

Abstract.

In this paper we introduce and discuss two proprieties related to recurrences in dynamical systems. The first gives the asymptotic law for the return time in a neighborhood, while the second gives a topological index of fractal type to characterize the system or some regions of the system.

Key-Words: Recurrence, Poisson process, Topological entropy.

Number of figures : 2

February 1998
CPT-98/P3623

anonymous ftp : ftp.cpt.univ-mrs.fr
web : www.cpt.univ-mrs.fr

*Unité Propre de Recherche 7061

‡talk given by Sandro Vaienti at the conference "Disorder and Chaos" (Rome 22-24th sept. 1997), in honour of Giovanni Paladin

[1]Centre de Physique Théorique, CNRS Luminy, Case 907, F-13288 Marseille - Cedex 9, FRANCE, and PhyMat, Université de Toulon et du Var. 83957 La Garde, FRANCE

1 Introduction

The study of recurrence is at the hearth of ergodic theory. The first rigorous result in this field is probably the famous Poincaré's recurrence theorem, which states that given the dynamical system (X, T, μ) where T is a measurable map on X and μ a $T-$invariant probability measure, then $\mu-$almost every point in each measurable subset $A \subset X$ comes back to A an infinite number of times.

Poincaré's result deals with a measurable recurrence propriety in the sense that the measure plays a fundamental role. But there are more genuine topological recurrence proprieties, like , for example, the density of all the orbits for irrational rotations. In this paper we present and discuss two problems which are related to these two aspects of recurrence, the measurable and the topological ones. We will get some precise statistics of the return time of typical orbits in a given neighborhood, while in the second case we will

get some global informations on the asymptotic distribution of orbits through the definition of a "dimension". The construction of this dimension uses explicitly the shortest return time in a given set and is a reminiscent of the way of constructing Haussdorf's dimension : this explains the attribute fractal in the title of our note, but probably there are deeper reasons. We hope that this dimension could be used as a statistical indicator of chaos and complexity.

We now define the fundamental quantities investigated in this paper. Take U a subset of X and define for each $x \in U$ the first return time into U as :

$$\tau_U(x) = \inf \left\{ k > 0 \,\middle|\, T^k x \in U \right\}.$$

The Poincaré recurrence of a *point* $\tau_U(x)$ as defined above leads to the first return time of a *set* : it is the infimum over all return tim es of the points of the set, and we denote it

$$\tau(U) = \inf_{x \in U} \tau_U(x).$$

2 Poisson statistics for the return time

We will consider as above a dynamical system (X, T, μ) where X is a (not necessarily compact) metric space, T a measurable application on X and μ a probability $T-$invariant Borel measure. A refinement of Poincaré's theorem can be found if we assume that μ is $T-$ergodic. Under this hypothesis, take U a measurable subset of X. Then the function $\tau_U : U \to \mathbb{N}$ is measurable. If we introduc e the induced probability measure μ_U defined by $\mu_U = \frac{\mu_{|U}}{\mu(U)}$, then the Kac Lemma [20] states that

$$\int_U \tau_U(x) \mathrm{d}\mu_U = \frac{1}{\mu(U)},$$

whenever $\mu(U) > 0$. Suppose now that $U_\varepsilon(z)$ is a neighborhood of $z \in X$ with diameter ε. Define the random variable $\mu(U_\varepsilon)\tau_{U_\varepsilon(z)}$ (we will call it the normalized return time). Next results assert that, under very general conditions, the normalized return time converges in law, with respect to the measure $\mu_\varepsilon \equiv \mu_{U_\varepsilon(z)}$, to a mean one exponential random variable, and that for $\mu-$almost every $z \in X$, precisely :

$$\mu_\varepsilon \Big\{ x \in U_\varepsilon \Bigm| \mu(U_\varepsilon)\tau_{U_\varepsilon}(x) > t \Big\} \underset{\varepsilon \to 0}{\longrightarrow} e^{-t}. \tag{1}$$

This kind of result was first proved for Axiom-A systems in a series of independent papers by Pitskel [22], Hirata [13], Collet [7]. Actually, these authors proved a stronger result : in fact they consider the sequence of successive normalized return times in τ_{U_ε} and proved that this sequence converges to the Poisson point process in finite-dimensional distribution when $\varepsilon \to 0$.

We want here to give the sketch of a proof which shows how naturally the exponential law for the statistic of the first return time arises.

Sketch of proof. Let $t > 0$ be fixed, $U \subset X$, $\tau = \tau(U)$ and $n = n(U) > 0$ which will be fi xed later. We want to estimate the quantity

$$\mu_U \Big\{ x \in U \Bigm| n(U)\tau_U(x) > t \Big\}. \tag{2}$$

This can be rewritten as

$$\mu_U(T^{-1}U^c \cap \cdots \cap T^{-t/n}U^c) = \mu_U(T^{-\tau}U^c \cap \cdots \cap T^{-t/n}U^c).$$

Now we suppose (H1) that τ is big enough, and that (H2) the events $T^{-\tau}U^c$, ..., $T^{-t/c}U^c$ are nearly independent, which yields

$$(2) \approx \mu(T^{-\tau}U^c \cap \cdots \cap T^{-t/n}U^c) \approx (1 - \mu(U))^{(t/n)-\tau}.$$

we assume now (H3) that $t/n(U) - \tau(U)$ tends to infinity as $\mu(U)$ goes to zero,

$$(2) \approx \exp\left(-\frac{\mu(U)}{n(U)}t + \mu(U)\tau(U)\right).$$

The exponential law appears now if we take the normalization $n(U) = \mu(U)$, and (H3) now reads $\mu(U)\tau(U) \to 0$ as $\mu(U)$ vanishes. □

Besides the statistics of return times one could equivalently study the process of the successive entrance times into a region Ω and prove under some conditions that, when times are correctly rescaled, the process converges in law to a Poisson point process [8]. In some cases it is even possible to estimate the rate of this convergence [12], and it is a promising field for further researches.

The Poisson statistics for the first return times was successively extended by Hirata [14] to a large class of systems verifying what he called the "self-mixing" conditions. Unfortunately these conditions do not hold for non-uniformly hyperbolic dynamical systems. Nevertheless the techniques introduced in [14] can be adapted to this kind of dynamical system. In particular we will consider the well known one parameter family of one-dimensional intermittent maps :

$$T(x) = \begin{cases} x(1 + 2^\alpha x^\alpha); & \forall x \in [0, 1/2[, \\ 2x - 1; & \forall x \in [1/2, 1]. \end{cases}$$

When the parameter $\alpha > 1$, T has a $\sigma-$finite absolutely continuous invariant measure : in these cases it is possible to prove that the sequence of successive (suitably normalized) entrance times in a small neighborhood of the neutral fixed point converges to a Poisson point process provided the system is equipped with an absolutely continuous distribution with density of bounded variations.

We instead consider the case $0 < \alpha < 1$, for which we construct in the paper [17] an absolute continuous invariant probability measure by using a new technique based on a stochastic perturbation of the Perron-Froebenius operator.

We also proved polynomial decay of correlations for Hölder continuous observab les. The case $0 < \alpha < 1$ is quite different if compared with the case of the $\sigma-$finite measure, especially for the techniques used to control the disto rtion of the application. Moreover, strictly speaking, our analysis will be local arou nd almost any point and not only restricted to a neighborhood of the neutral fixed point.

Let us now consider the infinite Markov partition ξ generated by the left preimages of 1 and consider the dynamical partitions $\xi_n \equiv \bigvee_{i=0}^{n-1} T^{-1}\xi$. If we now fix a point $z \in [0,1]$ and denote by U_n the member of ξ_n containing z and by $\mu_n \equiv \frac{\mu_{U_n}}{\mu(U_n)}$, we could ask if the propriety 1 follows. This is just the content of the next theorem.

Theorem 2.1. *For $\mu-$almost every $z \in [0,1]$ and $\forall t > 0$ we have :*

$$\mu_n \Big\{ x \in U_n \ \Big| \ \mu(U_n)\tau_{U_n} > t \Big\} \xrightarrow[n\to\infty]{} e^{-t}.$$

The proof of this theorem heavily relies on the techniques developed in [17] for decay of correlations. (Laplace transform technique are used in the rigorous proof).

3 Dimensional characteristic for Poincaré recurrence

In this section, we present a new approach originally introduced by Afraimovich [1] for minimal sets, in order to characterize topological recurrence. We will present a series of preliminary results, some of which apply to general dynamical systems, others to specific class of hyperbolic systems : the main result is the possibility to define a "dimension" (in the fractal language) which turns out to be an invariant for topological conjugation.

This dimension reveals to be a good indicator to distinguish systems of zero topological entropy. On the contrary, in the case of some positive entropy systems, it coincides with topological entropy (for subshifts of finite type and $\beta-$shifts for example). It is a matter of investigation whether the identification with topological entropy persists for more complicated chaotic systems. We will present some numerical evidence in this direction at the end of this chapter.

3.1 Construction of the dimension

We are using the well known Caratheodory's construction, for which Hausdorf's measures are special cases. We will work on a compact metric space

(X, d) together with a continuous transformation T, which form a dynamical system (X, T).

For any $A \subset X$, we define $\mathcal{R}^A(A, \varepsilon)$ the collection of all countable covers of A by subsets of X with diameters less than ε (the superscript A above $\mathcal{R}$ stands for *arbitrary* type of set that form the cover). In the same way, we denote by $\mathcal{R}^O$ and $\mathcal{R}^F$ the restriction of the precedent collection to cover with respectively *open* and *closed* sets.

Then we define a pre-measure (or gauge function) $\Phi(U) : 2^X \to \mathbb{R}^+$ with the propriety that $\Phi(\emptyset) = 0$.

Now we define [1]

$$M_\alpha^\Phi(A, \varepsilon) = \inf_{R \in \mathcal{R}(A,\varepsilon)} \sum_{U \in R} \Phi^\alpha(U). \tag{3}$$

(we do not precise here if we use covers by arbitrary sets, open sets or closed sets.) It is easy to show that M_α^Φ is a family of outer measure with the parameter α.

The idea of Afraimovich was to apply this construction in the case where $\Phi(U)$ is a decreasing function of $\tau(U)$, i.e. $\Phi(U) = g \circ \tau(U)$ where $g : \mathbb{N} \to \mathbb{R}$ is decreasing and converge to zero. We will also call this function a gauge function. Typically, we will set $\Phi(U) = e^{-\tau(U)}$ or $\Phi(U) = \frac{1}{\tau(U)}$, the choice being determined by the type of growth rate of Poincaré recurrence with respect to the diameter in our system. From now on, we will implicitly use one of these pre-measure.

Theorem 3.1 ([21]). *The outer measure for Poincaré recurrence $M_\alpha^{A,\Phi}$ constructed with arbitrary covers is concentrated on periodic points. The outer measures for Poincaré recurrence $M_\alpha^{O,\Phi}$ and $M_\alpha^{F,\Phi}$ constructed respectively with open and closed covers, coincide on* closed sets*.*

Remark. *We recall that in the case of Haussdorf's measures (obtained when we set $\Phi(U) = |U|$), the three constructions (arbitrary, open and closed covers) coincide on any borelian sets [23].*

The first part of this theorem results from a very simple construction, in which we construct covers of the space minus its periodic points with sets

[1] a slightly more general definition would be to, instead of put Φ^α in the sum, rather use a one parameter family of pre-measure Φ_α.

that all have infinite Poincaré recurrence [2] ! (but usually these sets cannot be borelian …) The second part is due to a nice propriety of the pre-measure Φ and comes from a more general theorem that we prove in [21].

This indicate that the choice of open or closed covers is natural, since otherwise we would obtain trivial results.

We now take the limit $\varepsilon \to 0$:

$$m_\alpha^\Phi(A) = \lim_{\varepsilon \to 0} M_\alpha^\Phi(A, \varepsilon). \tag{4}$$

The set function $m_\alpha^\Phi(A)$ is a family of *Borelian measure* (as shown in [10]).

If $\Phi(U)$ goes uniformly to zero when $|U|$ goes to zero [3] then we meet all the conditions of the classical Carathéodory's construction. It is well known, then, that there exists a unique transition exponent $\alpha_c^\Phi(A)$ such that

$$m_\alpha^\Phi(A) = \begin{cases} \infty & \text{if } \alpha < \alpha_c^\Phi(A) \\ 0 & \text{if } \alpha > \alpha_c^\Phi(A) \end{cases}$$

It is true with Haussdorf's measures, and Afraimovich has proved the same result with Poincaré recurrence if the system is minimal[4] [1].

However, we will consider cases where the transition point is not so net, and still we can define a critical exponent without any ambiguity (see Figure 1). This is possible because the set function $m_\alpha^\Phi(A)$ is non-increasing with α. So, we define the critical exponent of a set $A \subset X$ as

$$\alpha_c^\Phi(A) = \sup \left\{ \alpha > 0 \;\middle|\; m_\alpha^\Phi(A) = \infty \right\}. \tag{5}$$

It is always well defined and positive if we adopt the convention that $\sup \emptyset = 0$. We will call this dimension-like characteristic either dimension for Poincaré recurrence, either Afraimovich-Pesin (AP)'s dimension.

[2] To give an idea : suppose that the transformation T is invertible, then we can construct a set U by taking one point (and no more) of each non-periodic orbit of the system (remark that we need to use the Axiom of Choice to do that). One can check that $\tau(U) = \infty$, but even more : $\forall k, \tau(T^k U) = \infty$. Thus, the countable family of set $U_k \equiv T^k U$ is a cover of the space minus the periodic points, whose members have all infinite Poincaré recurrence. Because of Poincaré's recurrence theorem, these sets cannot have positive measure for any invariant measure, and thus for any invariant non-atomic probability measure they cannot be all measurable .

[3] more preci sely if $\forall \varepsilon > 0, \quad \exists \delta \quad \text{such that} \quad |U| < \delta \Rightarrow \Phi(U) < \varepsilon$.

[4] a dynamical system is minimal if each orbit is dense in the space.

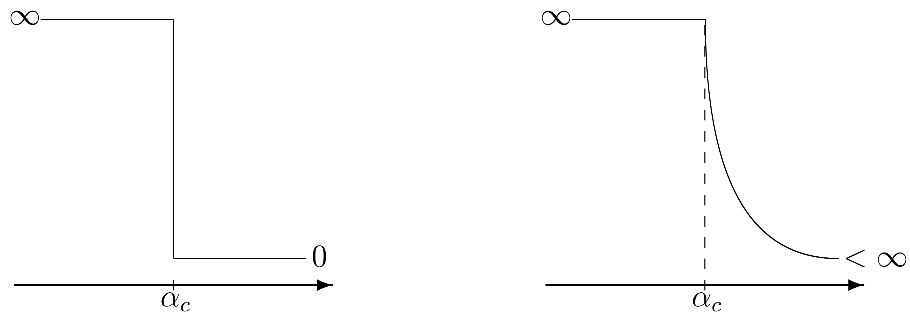

Figure 1. On the left, the transition is net, on the right it is not, however we are able to define a unique critical exponent α_c in both cases.

3.2 Some proprieties of Afraimovich-Pesin's dimension

Theorem 3.2. *The borelian measure m_α^Φ and the dimension α_c^Φ, constructed with open or closed covers, are invariant under topological conjugation, i.e. if (X, d, T) and (X', d', T') are two continuous dynamical system on metric spaces and if there exists a homeomorphism $h : X \to X'$ such that $T = h^{-1} \circ T' \circ h$, then for any $A \subset X$, $m_\alpha^\Phi(A) = m_\alpha'^\Phi(h(A))$.*

We would like to remark that this result is important since it allows us to say that AP's dimension is a *topological propriety*, i.e. two dynamical systems that are similar from a certain (topological) point of view will have the same AP's dimension.

Topological entropy is a tool that allows us to classify systems that are similar. The problem is that it cannot be used to distinguish systems with zero topological entropy although they may have very different behavior, which show the need to find other tools. AP's dimension with a non-exponential gauge function (e.g. $\Phi(U) = \frac{1}{\tau(U)}$) is such a tool.

It is interesting to note that for some classes of minimal sets with zero topological entropy, some topological invariant numbers have been recently proposed : for example the symbolic (or topological) complexity ţteferenczi, and the covering number [6]. It would be interesting to compare them to AP's dimension.

Proof of Theorem 3.2. We recall that we supposed from the beginning that X is compact, so that h is uniformly continuous. We write the uniform continuity

$$\forall \delta > 0, \exists \varepsilon(\delta) \text{ such that } \forall x, y \in X, \text{ if } d(x,y) < \varepsilon(\delta) \text{ then } d'(h(x), h(y)) < \delta.$$

Let $A \subset X$ and $A' = h(A) \subset X'$. Let $\mathcal{R}'(A', \delta)$ be the set of all covers R' of A' with sets of diameter less than δ. Let $h(\mathcal{R}(A, \varepsilon(\delta)))$ be the set of all transformed covers $h(R) \equiv \{h(U), U \in R\}$ of A with sets of diameter less than $\varepsilon(\delta)$. Then $\mathcal{R}'(A', \delta)$ contains $h(\mathcal{R}(A, \varepsilon(\delta)))$. Moreover, topological conjugation implies obviously that $\tau(U) = \tau'(h(U))$ for any set $U \subset X$, thus we have $\Phi(U) = \Phi'(h(U))$. This shows that

$$M_\alpha^\Phi(A, \varepsilon(\delta)) = \inf_{R \in \mathcal{R}(A, \varepsilon(\delta))} \sum_{U \in R} \Phi(U)^\alpha \geq \inf_{R' \in \mathcal{R}'(A', \delta)} \sum_{U' \in R'} \Phi'(U')^\alpha = M_\alpha'^{\Phi'}(A', \delta).$$

Then, taking the limit $\delta \to 0$ (hence $\varepsilon \to 0$), we obtain $m_\alpha^\Phi(A) \geq m_\alpha'^{\Phi'}(A')$. Now, by reversing A and A''s rules, one can apply the same idea to obtain the opposite inequality, which yields

$$m_\alpha^\Phi(A) = m_\alpha'^\Phi(A').$$

It is then obvious that $\alpha_c^\Phi(A) = \alpha_c'^{\Phi'}(A')$. □

The next theorem establishes other proprieties of AP's dimension, the most important being the one that says that AP's dimension over X coincide with AP's dimension restricted to the set of non-wandering points[5], that is exactly what happens for the topological entropy.

Theorem 3.3. *Dimension for Poincaré recurrence has the following proprieties :*

1. *if we use the pre-measure* $\Phi(U) = e^{-\tau(U)}$, *then for any* $k > 0$, *we have* $\alpha_c^\Phi(T^k, X) \leq k\alpha_c^\Phi(T, X)$,
2. *if we use the pre-measure* $\Phi(U) = \frac{1}{\tau(U)}$, *then for any* $k > 0$, *we have* $\alpha_c^\Phi(T^k, X) \leq \alpha_c^\Phi(T, X)$,
3. *if* T *is invertible, then* m_α^Φ *is an invariant measure.*
4. $\alpha_c^\Phi(T, X) = \alpha_c^\Phi(T, NW) = \alpha_c^\Phi(T_{|NW}, NW)$, *where* NW *denotes the set of non-wandering points.*

[5] a point x is non-wandering if any open ne ighborhood V has a finite Poincaré recurrence, i.e. $\tau(V) < \infty$.

5. *if we use the pre-measure $\Phi(U) = e^{-\tau(U)}$, there is the lower-bound*

$$\alpha_c^\Phi(X) \geq \overline{\lim}_{k\to\infty} \frac{1}{k} \log \#\mathrm{Per}(k),$$

where $\#\mathrm{Per}(k)$ denotes the number of periodic points with smallest period k.

Remark. *We point out that these results are true with open and closed covers. We proved it in [21]. There are examples of diffeomorphisms of the unit disk where strict inequality holds in the two first points of this theorem. However, these constructions depend heavily on the combinatorics of the periodic points, and these maps are somewhat unnatural. That is why we conjecture that fo r a large class of dynamical systems the equality holds. We recall that for topological entropy, the following equality holds :*

$$h_{\mathrm{top}}(T^k) = k h_{\mathrm{top}}(T).$$

The last point gives a lower-bound to AP's dimension with the periodic points. We recall that there exist a similar lower-bound for topological entropy with expansive map ([24], p178) :

$$h_{\mathrm{top}} \geq \overline{\lim}_{k\to\infty} \frac{1}{k} \log \#\mathrm{Fix}(k),$$

where $\#\mathrm{Fix}(k)$ denotes the number of fixed point of T^k.

3.3 Application of AP's dimension to classical dynamical systems

3.3.1 Systems with positive topological entropy

We now state some results about AP's dimension in simple cases, as subshifts of finite type and $\beta-$shift. For these systems, AP's dimension with exponential gauge function ($\Phi(U) = e^{-\tau(U)}$) and topological entropy are equal.

These systems are symbolic systems. We will work on the space $\Omega = \{0, \ldots, p-1\}^{\mathbb{N}}$ of all semi-infinite sequences $\omega = \omega_1\omega_2\ldots$, with the product topology. We consider the shift to the left σ such that $\sigma\omega_1\omega_2\omega_3\ldots = \omega_2\omega_3\ldots$. Subshifts of finite type and $\beta-$shift are restrictions of the shift on some invariant subsets of Ω. See [18, 19] for a complete description of these systems.

Many dynamical systems are topologically conjugate to subshifts of finite type, whereas the $\beta-$transformation $T_\beta(x) = \beta x \mod 1$ is conjugate to the $\beta-$shift [18].

We now state our results.

Theorem 3.4. *For subshifts of finite type and $\beta-$shifts, AP's dimension as defined above, with the pre-measure $\Phi(U) = e^{-\tau(U)}$, is equal to topological entropy.*

However, one can produce simple examples where there is not equality, the most trivial one being the identity transformation, for which topological entropy is zero whereas AP's dimension is infinite (because of the presence of an infinite number of fixed point), whichever gauge function one choose. Other examples are the one for which strict inequality holds in the first point of Theorem 3.3. We believe that topological entropy is a lower bound for AP's dimension (with exponential gauge function) and furthermore that equality holds for systems with strong chaotic proprieties.

3.3.2 Systems with zero topological entropy

AP's dimension for these systems is even more interesting to study, since it might let us to distinguish between them, whereas topological entropy cannot. We choose then a hyperbolic gauge function $\Phi(U) = \frac{1}{\tau(U)}$.

The examples studied by Afraimovich are irrational rotations on the circle and Denjoy example [1]. In these cases, he obtained a dimension following the diophantine characteristic of the rotation number. (However, we have to point out that for the moment these results are proved only if we restrict the covers to one by open *intervals*. Fortunately, in one dimension, the propriety "the set U is an open interval" is topologically invariant, so that AP's dimension defined with covers by intervals is still a topological invariant number.) This result is very interesting because Auslander and Katznelson [4] have proved that any transitive circle map without periodic points is topologically conjugate to an irrational rotation, and hence they will have the same AP's dimension.

3.4 Numerical results with the logistic map

The aim of this section is to show that it is possible and actually quite easy to perform numerical computation of AP's dimension. We computed the

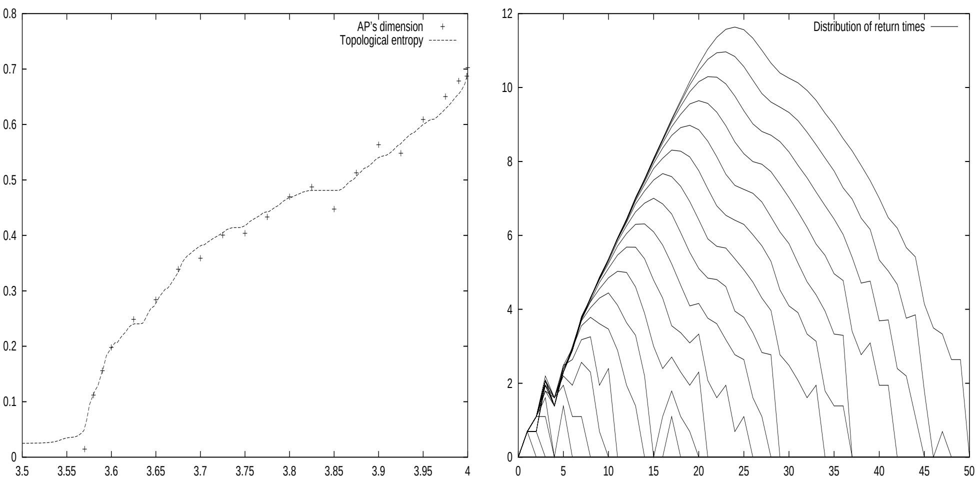

Figure 2. On the left, AP's dimension and topological entropy for logistic map $T_\mu(x) = \mu x(1-x)$. On the right, distribution of first return times for the same transformation with $\mu = 3.9$: it shows $\log N_\varepsilon(k)$ versus k for $\varepsilon = 2^{-n}, n = 1, 2, \ldots, 20$. We can see that $N_\varepsilon(k)$ tends to some limit distribution $N(k)$ from below.

dimension on map of the interval that is not conjugated to some subshifts of finite type so that the result is not trivial, but for which we have many reasons to believe that we have equality between AP's dimension and topological entropy.

We have chosen the well known logistic map : it is the transformation $T_\mu : \mathbb{R} \to \mathbb{R}$:

$$T_\mu(x) = \mu x(1 - x).$$

For any $\mu \geq 4$, it is topologically conjugated to the shift transformation on the space $\{0, 1\}^{\mathbb{N}}$, therefore we can affirm that AP's dimension in this case is $\alpha_c = h_{\text{top}} = \log 2$.

Thus we chose to consider some value $\mu < 4$. We point out that the logistic map is a continuous map of the interval with negative Schwartzian derivative, so it follows from a Theorem ([ALM], p.219-220) that

$$h_{\text{top}}(T_\mu) = \overline{\lim}_{k\to\infty} \frac{1}{k} \log \#\text{Fix}(k) = \overline{\lim}_{k\to\infty} \frac{1}{k} \log \#\text{Per}(k).$$

We now explain the technique that has been used to perform the calculation : Let's consider $\mathcal{R}_\varepsilon$ a cover of the space X with sets of diameter ε. We denote by $N_\varepsilon(k)$ the number of element U of this cover for which $\tau(U) = k$. It allows us to bound the sum (3) with

$$M_\alpha(X, \varepsilon) \leq \sum_{k=1}^{\infty} N_\varepsilon(k) e^{-\alpha k}.$$

If we note $N(k) = \sup_{\varepsilon>0} N_\varepsilon(k)$, then it is possible to prove that the critical exponent α_c is bounded by

$$\overline{\lim}_{k\to\infty} \frac{1}{k} \log \#\text{Per}(k) \leq \alpha_c \leq \overline{\lim}_{k\to\infty} \frac{1}{k} \log N(k).$$

The first inequality is actually always verified (provided T is continuous). Thus the numerical experimentation shows that for the logistic map $N(k) \propto \#\text{Per}(k)$ and therefore that AP's dimension and topological entropy coincide.

Acknowledgments

We want to thanks M. Hirata who is actually the coauthor of section two's result s and V. Afraimovich who introduced us to the recurrence dimension. We also thank S. Kolyada for remarks and stimulating discussions. S. Vaienti acknowledges the support of the ISI-Turin.

References

[1] V. Afraimovich, Chaos **7** (1997)

[2] V. Afraimovich, G.M. Zaslavsky, "Sticky Orbits of Chaotic Hamilt onian Dynamics", Preprint 1998

[3] L. Alsedà, J. Llibre and M. Misiurewicz, "Combinatorial Dynamics and Entropy in Dimension One", (1993, World Scientific Publ., Singapore)

[4] J. Auslander, Y. Katznelson, Israel J. Math, **32**, 375-381, (1979)

[5] M. Campanino, S. Isola, Forum Mathematicum, **7**, 331–34 8 (1995)

[6] N. Chabarchina, "Nombres de recouvrement", Thèse - I.M .L. Marseille (1997)

[7] P. Collet, Lectures given at the C.I.M.P.A. - Temuco (1991)

[8] P. Collet, A. Galves, Journal of Statistical Physics, **72**, 459–478 (1993)

[9] P. Collet, A. Galves, B. Schmitt, "Fluctuation of repetition times for Gibbsian sources", Preprint 1997

[10] K.J. Falconer, "The geometry of fractal sets", (1985, Cambridge University Press)

[11] S. Ferenczi, to appear in Israel Journal of Mathematics (1997)

[12] A. Galves, B. Schmitt, Random Comput. Dyn. to appear (1 997)

[13] M. Hirata, Ergodic Th. & Dynamical Systems, **13**, 533–55 6 (1993)

[14] M. Hirata, "Poisson law for the dynamical systems with the "self mixing" conditions", preprint TMU (1996)

[15] M. Hirata, B. Saussol, S. Vaienti, "Poisson law for the return time in an intermittent map", in preparation

[16] J.L. Knig, J. Analyse Math, **51**, 182–227 (1988)

[17] C. Liverani, B. Saussol, S. Vaienti, to appear in Ergodic Th. & Dynamical Systems (1997)

[18] W. Parry, Acta Math. Acad. Sci. Hung. **11**, 401–416 (1960)

[19] W. Parry, Trans. A.M.S., **122**, 368–378 (1966)

[20] K. Petersen, "Ergodic Theory", (1983, Cambridge University Press).

[21] V. Penné, B. Saussol, S. Vaienti, "A topological dynamical number constructed with recurrence times", preprint (1998)

[22] B. Pitskel, Ergodic Th. & Dynamical Systems, **11**, 501–5 33 (1991)

[23] C.A. Rogers, "Hausdorff measures", (1970, Cambridge)

[24] P. Walters, "Ergodic Theory - Introductory Lectures", (1975, Springer-Verlag)