LARGE DEVIATION FOR RETURN TIMES

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ABSTRACT. We prove a large deviation result for return times of the orbits of a dynamical system in a r-neighbourhood of an initial point x. Our result is a differentiable version of the work by Jain and Bansal who considered the return time of a stationary and ergodic process defined in a space of infinite sequences.

Keywords: Return time, exponential rate, conformal repeller, large deviation.

1. Introduction

Consider a dynamical system $(X, \mathcal{A}, g, \mu)$ where X is a compact metric space, $\mathcal{A}$ is a σ-algebra on X, $g : X \to X$ is a measurable map and μ an invariant probability measure on $(X, \mathcal{A})$. Let $A \subset X$ be a measurable set of positive measure. An important result in ergodic theory is the Poincaré's recurrence theorem. It states that any probability measure preserving map has almost everywhere recurrence. More precisely, for μ-almost every $x \in A$ we have that $\{n : g^n x \in A\}$ is infinite. It is natural to ask for quantitative results of the recurrence. Given a point $x \in A$, the first return time of the orbit of x to the set A is given by

$$\tau_A(x) = \min \{n \geq 1 : g^n x \in A\}.$$

In [10], Kac was able to prove that, when the system is ergodic, the expectation of the return time in a set A is equal to the inverse of the measure of this set, i.e.,

$$\int_A \tau_A d\mu = 1.$$

Moreover, if μ is not ergodic, the inequality

$$\int_A \tau_A d\mu \leq 1,$$

still holds.

In the study of quantitative recurrence is investigated properties related to the return time $\tau_r(x)$ that is defined as follows: for every $x \in X$, the return time of x under the map g in its r-neighborhood as

$$\tau_r(x) = \tau_{B(x,r)}(x) = \min\{n \geq 1 : d(g^n x, x) < r\}.$$

In dynamical system this subject have been studied by many authors. Boshernitzan [3] has studied rate of recurrence when X has a finite Hausdorff dimension. In symbolic space, Feng and Wu [7] investigated the set

$$E(\alpha, \beta) := \left\{ x \in [0,1] : \underline{\lim}_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = \alpha, \overline{\lim}_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = \beta \right\}$$

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and showed that for $0 \leq \alpha \leq \beta \leq \infty$, $\dim_H E(\alpha, \beta) = 1$. Analogous results are proved in [17] in the case of $C^{1+\alpha}$ conformal repellers in smooth compact manifolds and in [12] to the Gauss transformation on $[0, 1)$. It is important to emphasize that if we consider a repeller $J \subset M$, M a smooth manifold, the spectrum for recurrence $\mathcal{F}(\alpha) = \dim_H E(\alpha, \alpha)$ is degenerate, that is, $\mathcal{F}(\alpha) = \dim_H J$.

In [16], the author reviews important results on recurrence. He considers an expanding map of the interval and proves results for recurrence rates, limiting distributions of return times, and short returns. In [8] was presented an upper bound for the exponential approximation of the law of a hitting time in a mixing dynamical system.

The case of large deviation has been investigated by Abadi and Vaienti in [1]. It was proved large deviation properties of $\tau(C_n)/n$, where $\tau(C_n)$ is the first return of a n-cylinder to itself. More precisely, if the system is ψ-mixing, if $\psi(0) < 1$ and the Rényi entropies exist for all integers β, then for $\delta \in (0, 1]$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \mu \{x : \tau(C_n) \leq [\delta n]\} := M(\delta)$$

exists. In addition, they explicit the form of $M(\delta)$. If we denote with $\tau_A^n(x)$ the nth return times into A, the Birkhoff theorem gives that for μ-almost every point x

$$\lim_{n\to\infty} \frac{\tau_A^n(x)}{n} = \frac{1}{\mu(A)}.$$

For Axiom A diffeomorphisms and equilibrium states μ, it was proved by Chazottes and Leplaideur [5] and after by Leplaideur and Saussol [11], under the assumption that $\mu(\partial A) = 0$, the existence of a rate function Φ_A such that for every $u \geq \frac{1}{\mu(A)}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mu \left\{ \frac{\tau_A^n}{n} \geq u \right\} = \Phi_A(u)$$

and for every $0 \leq u \leq \frac{1}{\mu(A)}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mu \left\{ \frac{\tau_A^n}{n} \leq u \right\} = \Phi_A(u).$$

Our result is a differentiable version of a recent work by Jain and Bansal [9] who studied large deviation property for normalized version of recurrence times under ϕ-mixing conditions. Let H denote the entropy rate of the process X and x a particular realization of X. Define the first return time of x_1^n as

$$R_n(x) = \min \left\{ j \geq 1 : x_1^n = x_{-j+1}^{-j+n} \right\}.$$

We say that X have exponential rates for entropy if for every $\epsilon > 0$, we have

$$P\left(\left\{ x_1^n : 2^{-n(H+\epsilon)} \leq P(x_1^n) \leq 2^{-n(H-\epsilon)} \right\} \right) \leq 1 - r(n, \epsilon),$$

where $r(\epsilon, n) = e^{-k(\epsilon)n}$, with $k(\epsilon)$ a real valued positive function of ϵ. They proved that for an exponentially ϕ-mixing process with exponential rates for entropy,

$$P\left(\left| \frac{\log R_n(X)}{n} - H \right| > \epsilon \right) \leq 2e^{-I(\epsilon)n}, \forall n \geq M(\epsilon),$$

where $I(\epsilon)$ is a real positive valued function for all $\epsilon > 0$ and $I(0) = 0$.

If we denote with

$$\underline{d}_\mu(x) = \varliminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \varlimsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

the lower and upper pointwise dimensions of the measure μ at the point $x \in X$, it was proved by Barreira and Saussol [2] that

$$\varliminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \leq \underline{d}_\mu(x) \quad \text{and} \quad \varlimsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \leq \overline{d}_\mu(x),$$

for μ-almost every $x \in X$. If the system has a super-polynomial decay of correlations, Saussol in [15] showed that equalities will hold for the expressions above. This implies that

$$\log \tau_r(x) \underset{r\to 0}{\sim} \log\left(r^{-d_\mu(x)}\right).$$

Our aim is to study the limiting behavior as $r \to 0$ of $\mu\left(\tau_r \geq r^{-d_\mu-\epsilon}\right)$ and $\mu\left(\tau_r \leq r^{-d_\mu+\epsilon}\right)$. This characterization is via asymptotic lower exponential bound. We consider the limits

$$\varliminf_{r\to 0} \frac{1}{\log r} \log \mu\left(\tau_r \geq r^{-d_\mu-\epsilon}\right) \text{ and } \varliminf_{r\to 0} \frac{1}{\log r} \log \mu\left(\tau_r \leq r^{-d_\mu+\epsilon}\right)$$

and under some assumption about the measure we prove large deviation estimates (see Theorem 2.4 for precise statements). The bounds we prove are given in terms of the minimum of rate for dimension and rate for fast times (see Section 2 for definition). Moreover, we consider a $C^{1+\alpha}$ conformal repeller and an equilibrium state of a Hölder potential. Then, we compute the rates functions and applies Theorem 2.4 to obtain large deviation estimates for return times for repeller (see Theorem 3.2).

2. Large deviation estimates for return times in a general setting

Let $g : X \to X$ be a measurable map and μ an invariant probability measure on $(X, \mathcal{A})$.

Definition 2.1. *The measure μ is called exact dimensional if there exists a constant d_μ such that*

$$\underline{d}_\mu(x) = \overline{d}_\mu(x) = d_\mu \quad \textit{for} \quad \mu\textit{-almost every} \quad x \in X.$$

We recall that the Hausdorff dimension of a probability measure μ on X is given by

$$\dim_H \mu = \inf\{\dim_H Z : \mu(Z) = 1\},$$

where $\dim_H Z$ denotes the Hausdorff dimension of Z.

Moreover, for an exact dimensional measure, the Hausdorff dimension and the local dimension coincide:

Proposition 2.2 ([19]). *If μ is exact dimensional, then*

$$d_\mu = \dim_H \mu.$$

We now define the rates functions which will appear in our large deviations estimates:

Definition 2.3. *Given $\epsilon > 0$, we define the exponential rate for dimension:*

$$\underline{\psi}(\pm\epsilon) = \varliminf_{r\to 0} \frac{1}{\log r} \log \mu\left(\left\{\frac{\log \mu(B(x,r))}{-\log r} \in I_{\pm\epsilon}\right\}\right), \tag{1}$$

where $I_\epsilon = (-\infty, -d_\mu - \epsilon)$ and $I_{-\epsilon} = (-d_\mu + \epsilon, +\infty)$.

Given $\epsilon, a > 0$, we define the exponential rate for fast return times:

$$\underline{\varphi}(a,\epsilon)=\varliminf_{r\to 0}\frac{1}{\log r}\log\mu\left(\left\{x_0:\mu_{B(x_0,2r)}\left(\tau_{B(x_0,2r)}\le r^{-d_\mu+\epsilon}\right)\ge r^a\right\}\right). \tag{2}$$

We may now state our main result.

Theorem 2.4. *Let $(X,\mathcal{A},g,\mu)$ be a dynamical system. Suppose that μ is an exact dimensional measure. Given positive constants ϵ,ξ and a, for all $\gamma\in(0,1)$ we have:*

$$\underline{f}(\epsilon):=\varliminf_{r\to 0}\frac{1}{\log r}\log\mu\left(\tau_r\ge r^{-d_\mu-\epsilon}\right)\ge\min\left\{(1-\gamma)\epsilon,\underline{\psi}(\gamma\epsilon)\right\} \tag{3}$$

and

$$\begin{aligned}\underline{f}(-\epsilon) &:= \varliminf_{r\to 0}\frac{1}{\log r}\log\mu\left(\tau_r\le r^{-d_\mu+\epsilon}\right) \\ &\ge \min\left\{-\gamma(\epsilon+\xi)+a,\underline{\psi}(\gamma\epsilon),\underline{\varphi}(a,\epsilon),\underline{\psi}(-\gamma\xi)\right\}.\end{aligned} \tag{4}$$

This result is satisfactory in the sense that it holds for any dynamical system. We can observe that in (3) if the rate function for dimension $\underline{\psi}$ is positive in some interval $(0,\epsilon)$, it means that $\mu\left(\tau_r\ge r^{-d_\mu-\epsilon}\right)$ has a fast exponential decay.

For several dynamical systems, including Hénon maps (see [4]), we obtain the following result.

Proposition 2.5. *If there exist constants $a,b>0$ such that for all $r\in(0,1)$:*

- *there exists a set Ω_r such that*
$$\mu(\Omega_r^c)<r^b;$$
- *for all $x\in\Omega_r$,*
$$\left|\mu_{B(x,r)}\left(\tau_{B(x,r)}>\frac{t}{\mu(B(x,r))}\right)-e^{-t}\right|\le r^a,$$
for every $t>0$.

Then, $\varphi(a,\epsilon)\ge\min\{\psi(a-\epsilon),b\}$.

The proof of this proposition will be done at the end of Section 4.

3. Large deviation estimates for return times for conformal repeller

In this section we will present a version of our main result for conformal repeller. Let $\phi:X\to\mathbb{R}$ be a Hölder continuous function. We call Gibbs measure for the potential ϕ an invariant measure μ such that there exists a constant $P_g(\phi)\in\mathbb{R}$ such that for some $\kappa_\phi\ge 1$, for any x and n, we have

$$\frac{1}{\kappa_\phi}\le\frac{\mu(\mathcal{J}_n(x))}{exp(S_n\phi(x)-nP_g(\phi))}\le\kappa_\phi,$$

where $\mathcal{J}_n(x)$ is the cylinder of length n containing x.

Let $g:M\to M$ be a $C^{1+\alpha}$ map of a smooth manifold and consider a g-invariant compact set $J\subset M$. The map g is said to be an expanding map on J if there exist constants $c>0$ and $\beta>1$ such that

$$\|d_xg^nv\|\ge c\beta^n\|v\|$$

for every $n \in \mathbb{N}$, $x \in J$ and $v \in T_xM$. In addition, we call J a repeller if there exists an open neighborhood V of J such that

$$J = \bigcap_{n\geq 0} g^{-n}V.$$

The map g is said to be conformal on J if

$$d_x g = a(x) Isom_x,$$

where $Isom_x$ denotes an isometry of the tangent space T_xM.

From now on, let (J, g) be a conformal repeller. We collect some facts about HP-spectrum for dimensions.

Let (Σ_A^+, σ) be a subshift of a finite type and $\chi : \Sigma_A^+ \to J$ a coding map such that $\chi \circ \sigma = g \circ \chi$. Let μ be a Borel probability measure on a metric space (X, ρ). Denote by ζ a Hölder continuous function on J and $\mu = \mu_\zeta$ the equilibrium measure for (g, ζ). Moreover, let $\varphi = \zeta \circ \chi$ be a Hölder continuous function on Σ_A^+ and $\nu = \nu_\varphi$ its Gibbs measure. Finally, consider a function ψ such that $\log \psi = \varphi - P(\varphi)$.

Proposition 3.1 ([14]). *For any $q > 1$, the following limit exists*

$$HP_\mu(q) = \frac{1}{1-q} \lim_{r\to 0} \frac{\log \int \mu(B(x,r))^{q-1}}{-\log r} d\mu(x). \tag{5}$$

In addition, given $q \in (-\infty, \infty)$, let ϕ_q on Σ_A^+ be the one parameter family of functions such that

$$\phi_q(w) = -T(q) \log |a(\chi(w))| + q \log \psi(w),$$

where $T(q)$ is chosen such that $P(\phi_q) = 0$. For any $q > 1$,

$$\frac{T(q)}{1-q} = HP_\mu(q).$$

The function $T(q)$ is real analytic for all $q \in \mathbb{R}$, $T(0) = \dim_H J$, $T(1) = 0$, $T'(q) \leq 0$ and $T''(q) \geq 0$. And $T''(q) > 0$ if and only if the function $\log \psi - T'(q) \log |a(\chi(w))|$ is not cohomologous to a constant. If and only if μ is not a measure of maximal dimension.

Note that μ is exact dimensional (one can see [13] for more details).

Under this context, if we consider a conformal repeller and an equilibrium state of a Hölder potential ζ we obtain a version of our principal result, somewhat more concrete:

(1) in this setting we can compute the exponential rate for the dimension $\underline{\psi}$, using thermodynamic formalism;
(2) we can also estimate the exponential rate for fast return times $\underline{\varphi}$, using a technique similar to the one used to prove exponential return time statistics.

Thus, applying our main result to this setting will give us the following theorem.

Theorem 3.2. *Let (J, g) be a conformal repeller and μ an equilibrium state for a Hölder potential ζ. Given positive constants ϵ, ξ and a_0, for all $\gamma \in (0, 1)$, we have:*

(1)

$$\underline{f}(\epsilon) \geq \min \{(1-\gamma)\epsilon, \Lambda^*(-d_\mu + \gamma\epsilon)\} > 0; \tag{6}$$

(2)

$$\underline{f}(-\epsilon) \geq \min\{-\gamma(\epsilon+\xi)+a_0, \Lambda^*(-d_\mu+\gamma\epsilon), \Lambda^*(-d_\mu-\gamma\xi)\} > 0; \tag{7}$$

where $\Lambda^*(x) = -x + T^*(x) = -x + \sup_{\lambda\in\mathbb{R}}\{\lambda x - T(\lambda)\}$.

Remark 3.3. *If μ is the measure of maximal dimension the above theorem remains valid. However, since $HP_\mu(q)$ is constant equal to d_μ follows that $\Lambda^*(x) = +\infty$ for $x \neq 0$, which makes the minimum in* (6) *and* (7) *being $(1-\gamma)\epsilon$ and $-\gamma(\epsilon+\xi)+a_0$, respectively.*

In what follows we shall assume that there is no cohomology relation.

To obtain Theorem 3.2, we need a fundamental theorem of large deviation theory, the Gartner-Ellis Theorem.

Let μ_r be a family of probability measures. Consider a family $Z_r \in \mathbb{R}, r \in (0,1)$ where Z_r possesses the law μ_r and logarithmic moment generating function

$$\Lambda_r(\lambda) = \log \mathbb{E}\left[e^{\lambda Z_r}\right].$$

μ_r may satisfy the large deviation property if there exists a limit of properly scaled logarithmic moment generating functions.

Assumption 3.4. *For any $\lambda \in \mathbb{R}$, the logarithmic moment generating function, defined as the limit*

$$\Lambda(\lambda) := \lim_{n\to\infty} \frac{1}{-\log r}\Lambda_r(-\log r\lambda),$$

exists as an extended real number. Further, the origin belongs to the interior of the interval $D_\lambda := \{\lambda \in \mathbb{R}; \Lambda(\lambda) < \infty\}$, Λ is C^2 and strictly convex.

The Fenchel-Legendre transform of $\Lambda(\lambda)$ is

$$\Lambda^*(x) = \sup_{\lambda\in\mathbb{R}}\{\lambda x - \Lambda(\lambda)\}.$$

Remark 3.5. Λ^* *is strictly convex and C^1 on its support.*

Thus, we can enunciate Gartner-Ellis Theorem (see e.g. [6]).

Theorem 3.6 (Gartner-Ellis)**.** *If assumption 3.4 hold, then for any closed set F,*

$$\overline{\lim_{r\to 0}} \frac{1}{-\log r} \log \mu_n(F) \leq -\inf_{x\in F}\Lambda^*(x). \tag{8}$$

and for any open set G,

$$\underline{\lim_{r\to 0}} \frac{1}{-\log r} \log \mu_n(G) \geq -\inf_{x\in G}\Lambda^*(x). \tag{9}$$

We will apply this theorem to the family Z_r defined by

$$Z_r = \frac{\log \mu(B(x,r))}{-\log r}.$$

It follows that

$$\Lambda_r(\lambda) = \log \int e^{\lambda \frac{\log \mu(B(x,r))}{-\log r}} d\mu(x).$$

Thus, from the definition of $\Lambda(\lambda)$, we get

$$\begin{aligned}\Lambda(\lambda) &= \lim_{r\to 0}\frac{1}{-\log r}\log\int e^{\lambda\log\mu(B(x,r))}d\mu(x)\\ &= \lim_{r\to 0}\frac{1}{-\log r}\log\int \mu(B(x,r))^{\lambda}d\mu(x).\end{aligned}$$

The proof of the following proposition is an immediate consequence of the Proposition 3.1.

Proposition 3.7. *Let (J,g) be a conformal repeller and μ an equilibrium state for the Hölder potential ζ. Then, for $\lambda>0$, the following limit exists*

$$\Lambda(\lambda)=\lim_{r\to 0}\frac{1}{-\log r}\log\int \mu(B(x,r))^{\lambda}d\mu(x)=T(\lambda+1).$$

Applying Gartner-Ellis Theorem, we obtain:

Corollary 3.8. *Under the same conditions as in Proposition 3.7 we have that for all interval I,*

$$\lim_{r\to 0}\frac{1}{-\log r}\log\mu\left(\left\{\frac{\log\mu(B(x,r))}{-\log r}\in I\right\}\right)=-\inf_{x\in I}\Lambda^*(x),$$

where $\Lambda^*(x)=-x+T^*(x)$ is continuous on its domain.

Proof. This equality is a direct consequence of the Theorem 3.6. Since the logarithmic moment generating function is defined by $\Lambda(\lambda)=T(\lambda+1)$, the Fenchel-Legendre transform of $\Lambda(\lambda)$ is

$$\begin{aligned}\Lambda^*(x) &= \sup_{\lambda\in\mathbb{R}}\{\lambda x-\Lambda(\lambda)\}\\ &= \sup_{\lambda\in\mathbb{R}}\{\lambda x-T(\lambda+1)\}\\ &= \sup_{\nu\in\mathbb{R}}\{(\nu-1)x-T(\nu)\}\\ &= -x+\sup_{\nu\in\mathbb{R}}\{\nu x-T(\nu)\}\\ &= -x+T^*(x).\end{aligned}$$

The continuity of $\Lambda^*(x)$ follows from its convexity. □

In Figure 1, one can see a graph of the Frenchel-Legendre transform of Λ.

Figure 1. Graph of Λ^*.

Now, one can use Corollary 3.8, to get the rate function for the dimension:

Proposition 3.9. *For any $\epsilon>0$, the exponential rate for the dimension is given by:*

$$\underline{\psi}(\epsilon)=\inf_{x\in(-\infty,-d_\mu-\epsilon)}\Lambda^*(x)=\Lambda^*(-d_\mu-\epsilon)>0$$

and

$$\underline{\psi}(-\epsilon)=\inf_{x\in(-d_\mu+\epsilon,+\infty)}\Lambda^*(x)=\Lambda^*(-d_\mu+\epsilon)>0.$$

Proof. Recall that the exponential rate for dimension $\underline{\psi}$ is defined by

$$\underline{\psi}(\pm\epsilon) = \underline{\lim}_{r\to 0}\frac{1}{\log r}\log\mu\left(\left\{\frac{\log\mu(B(x,r))}{-\log r}\in I_{\pm\epsilon}\right\}\right), \tag{10}$$

where $I_\epsilon = (-\infty, -d_\mu - \epsilon)$ and $I_{-\epsilon} = (-d_\mu + \epsilon, +\infty)$. So, by Corollary 3.8 we have that

$$\begin{aligned}\underline{\psi}(\pm\epsilon) &= \underline{\lim}_{r\to 0}\frac{1}{\log r}\log\mu\left(\left\{\frac{\log\mu(B(x,r))}{-\log r}\in I_{\pm\epsilon}\right\}\right)\\ &= \inf_{x\in I_{\pm\epsilon}}\Lambda^*(x)\\ &= \Lambda^*(-d_\mu \mp \epsilon)\end{aligned}$$

which proves the proposition. □

From now on, assume that ζ is a potential such that $P(\zeta) = 0$. To obtain exponential rate for the return times (Prop. 3.13), we first review the shadowing property for periodic points.

Definition 3.10. *Given $\alpha > 0$, we call α-pseudo-orbit for (J, g) each sequence $(x_n)_{n\geq 0}$ such that*

$$d(gx_n, x_{n+1}) < \alpha, \text{ for all } n \geq 0.$$

We call a sequence $x_0, x_1, \cdots, x_{m-1}, x_m = x_0$ an α-periodic orbit if $d(gx_n, x_{n+1}) < \alpha$.

A particular case of an α-periodic orbit is provided by $x_0, gx_0, \cdots, g^{m-1}x_0$ when $d(g^m x_0, x_0) < \alpha$.

Lemma 3.11 (Shadowing lemma)**.** *If (J, g) is a repeller then for every $\beta > 0$ there exists $\alpha > 0$ such that given an α-pseudo-orbit $(x_n)_{n\geq 0}$ in J there exists $z \in J$ such that its orbit β-shadows $(x_n)_{n\geq 0}$, that is, $d(g^n z, x_n) < \beta$ for all $n \geq 0$.*

The proof can be seen in [18].

Lemma 3.12 (Closing lemma)**.** *If (J, g) is a repeller then for every α-periodic orbit $(x_n)_{n\geq 0}$ there exists a point y with $g^n(y) = y$ and $d(g^k y, x_k) < \beta$, for all $k = 0, \cdots, m-1$.*

We will use these properties to have information on the rate function for the return times:

Proposition 3.13. *Given $\epsilon, a_0 > 0$, the exponential rate for fast return times satisfies:*

$$\underline{\varphi}(a_0, \epsilon) \geq a_0 > 0.$$

This proposition is a simple consequence of the following lemma.

Lemma 3.14. *For any $d_0 \in (0, d_\mu)$ there exist constants $a_0, c_0, h > 0$ and a set Ω_r such that*

$$\mu(\Omega_r^c) < r^{a_0}$$

and for all $x_0 \in \Omega_r$, one has

$$\mu_{B(x_0,2r)}\left(\tau_{B(x_0,2r)} \leq r^{-d_0}\right) \leq 2k_\zeta(2r)^{hc_0} + r^{-d_0}k_\zeta\mu(B(x_0, 2r)).$$

Proof. We first claim that there exists Ω_r with $\mu(\Omega_r^c) \leq r^{a_0}$ such that for all $x_0 \in \Omega_r$ and for all $k \leq c_0 \log\frac{1}{2r}$ we have $B(x_0, 2r) \cap g^{-k}(B(x_0, 2r)) = \emptyset$.

Indeed, let $c_0 \in (0, d_\mu/\log m)$, where m is the number of branches of the map g. If x_0 is such that $B(x_0, 2r)\cap g^{-k}(B(x_0, 2r)) \neq \emptyset$, there exists x such that $d(x, x_0) < 2r$ and $d(g^k x, x_0) < 2r$, thus $d(x, g^k x) < 4r$. Take $4r = \alpha$ and consider the sequence $x, gx, \cdots, g^{k-1}x$. By the Closing lemma, there exists a point z such that $g^k z = z$ and $d(g^j z, g^j x) < \beta$ for all $j < k$. Moreover, in the proof of the Shadowing lemma one can see that we can take $\beta = c_1\alpha$, where $c_1 > 0$ depends only on the expansiveness constant. Define

$$\mathcal{P}_k = \{z : g^k z = z\} \text{ and } \mathcal{D}_k = \partial\mathcal{J}_k \supset \ disc(g^k),$$

where $disc(g^k)$ is the set of discontinuity points of g^k. It follows that

$$d(x, \mathcal{P}_k) < 4c_1 r \text{ or } d(x, \mathcal{D}_k) < 4c_2 r.$$

Therefore, $d(x, \mathcal{P}_k \cup \mathcal{D}_k) < c_3 r$. Observe that

$$\mathcal{C}_r(k) = \left\{x_0 : B(x_0, 2r) \cap g^{-k}(B(x_0, 2r)) \neq \emptyset\right\} \subset B(\mathcal{P}_k \cup \mathcal{D}_k, c_3 r) := \bigcup_{y \in \mathcal{P}_k \cup \mathcal{D}_k} B(y, c_3 r). \quad (11)$$

Moreover, using (17), we have the inequality

$$\begin{aligned} \mu\left(A_{(-\xi)}(2c_3 r) \cap B(\mathcal{P}_k \cup \mathcal{D}_k, c_3 r)\right) &\leq (\#\mathcal{P}_k + \#\mathcal{D}_k) \sup_{x \in A_{(-\xi)}(2c_3 r)} \mu(B(x, 2c_3 r)) \\ &\leq 2m^k (2c_3 r)^{d_\mu - \xi}. \end{aligned}$$

Take $K = c_0 \log \frac{1}{2r}$ and define

$$\Omega_r = A_{(-\xi)}(2c_3 r) \cap \bigcap_{k \leq K} B(\mathcal{P}_k \cup \mathcal{D}_k, c_3 r)^c.$$

It follows from the previous inequality and Corollary 3.8 that

$$\begin{aligned} \mu(\Omega_r^c) &\leq \sum_{k=1}^{K} \mu\left(A_{(-\xi)}(2c_3 r) \cap B(\mathcal{P}_k \cup \mathcal{D}_k, c_3 r)\right) + \mu\left(A^c_{(-\xi)}(2c_3 r)\right) \\ &\leq 2(2c_3 r)^{d_\mu - \xi} \sum_{k=1}^{K} m^k + (2c_3 r)^{\Lambda^*(-d_\mu + \xi) - \delta}, \\ &\leq 2(2c_3 r)^{d_\mu - \xi} m^{K+1} + (2c_3 r)^{\Lambda^*(-d_\mu + \xi) - \delta} \\ &\leq r^{a_0}, \end{aligned}$$

where $\delta > 0$ and $a_0 = \min\{d_\mu - \xi - c_0 \log m, \Lambda^*(-d_\mu + \xi) - \delta\}$. We observe that $x_0 \in \Omega_r$ implies that $x_0 \notin B(\mathcal{P}_k \cup \mathcal{D}_k, c_3 r)$. Therefore from (11), $B(x_0, 2r) \cap g^{-k}(B(x_0, 2r)) = \emptyset$ for all $k \leq c_0 \log \frac{1}{2r}$ which proves our initial claim.

We will now estimate the quantity $\mu_{B(x_0,2r)}\left(g^{-k} B(x_0, 2r)\right)$ for large values of k.

Let ζ be a Hölder potential. Recall that the Ruelle-Perron-Frobenius operator $\mathcal{L}_\zeta : C(M) \to C(M)$ is defined by

$$\mathcal{L}_\zeta(f)(x) = \sum_{y \in g^{-1}(x)} e^{\zeta(y)} f(y),$$

for $f \in C(M)$ and $x \in M$. By induction, for every $n \geq 1$,

$$\mathcal{L}^n_\zeta(f)(x) = \sum_{y \in g^{-n}(x)} e^{S_n \zeta(y)} f(y),$$

where $S_n\zeta = \sum_{k=0}^{n-1} \zeta \circ g^k$. Now we have that

$$\begin{aligned}
\mu\left(B(x_0,2r)\cap g^{-k}B(x_0,2r)\right) &= \int \mathbb{1}_{B(x_0,2r)}\mathbb{1}_{B(x_0,2r)}\circ g^k d\mu \\
&= \int \mathcal{L}^k(\mathbb{1}_{B(x_0,2r)})\mathbb{1}_{B(x_0,2r)} d\mu \\
&\leq \mu(B(x_0,2r)) \left\|\mathcal{L}^k(\mathbb{1}_{B(x_0,2r)})\right\|_\infty .
\end{aligned}$$

Hence,

$$\mu_{B(x_0,2r)}\left(g^{-k}B(x_0,2r)\right) \leq \left\|\mathcal{L}^k(\mathbb{1}_{B(x_0,2r)})\right\|_\infty . \tag{12}$$

It follows that for all $R \in \mathcal{J}_k$,

$$\begin{aligned}
\mathcal{L}^k(\mathbb{1}_R)(x) &= \sum_{y\in g^{-k}x,\ y\in R} e^{S_k\zeta(y)}\mathbb{1}_R(y) \\
&\leq \sum_{y\in g^{-k}x,\ y\in R} k_\zeta \mu(R),
\end{aligned}$$

where the last inequality follows from the Gibbs property since $P(\zeta) = 0$. In addition, the preimage of x under g^k has just one element in R. Then,

$$\mathcal{L}^k(\mathbb{1}_R)(x) \leq k_\zeta \mu(R). \tag{13}$$

If $c_0 \log \frac{1}{2r} < k < b_1 \log \frac{1}{2r}$, the ball $B(x_0, 2r)$ is contained in at most two cylinders. So there exists $h > 0$ such that

$$\begin{aligned}
\mathcal{L}^k(\mathbb{1}_{B(x_0,2r)}) &= \sum_{R\in\mathcal{J}_k} \mathcal{L}^k(\mathbb{1}_R) \\
&\leq 2k_\zeta\mu(R) \\
&\leq 2k_\zeta e^{-hk}.
\end{aligned}$$

Hence, we get

$$\mu_{B(x_0,2r)}(g^{-k}B(x_0,2r)) \leq 2k_\zeta e^{-hk}.$$

We now consider $R_L, R_R \in \mathcal{J}_k$ such that L, R are endpoints of the $B(x_0, 2r)$. If $k \geq b_1 \log \frac{1}{2r}$, we have that $\mu(R_L) \leq e^{-hk}$ and $\mu(R_R) \leq e^{-hk}$. Therefore, from (13) we have

$$\begin{aligned}
\mathcal{L}^k(\mathbb{1}_{B(x_0,2r)}) &= \sum_{R\in\mathcal{J}_k, R\subset B(x_0,2r)} \mathcal{L}^k(\mathbb{1}_R) + \mathcal{L}^k(\mathbb{1}_{R_L}) + \mathcal{L}^k(\mathbb{1}_{R_R}) \\
&\leq \sum_{R\in\mathcal{J}_k, R\subset B(x_0,2r)} k_\zeta\mu(R) + 2k_\zeta e^{-hk} \\
&\leq k_\zeta\mu(B(x_0,2r)) + 2k_\zeta e^{-hk}.
\end{aligned}$$

This implies

$$\mu_{B(x_0,2r)}\left(g^{-k}B(x_0,2r)\right) \leq k_\zeta\mu(B(x_0,2r)) + 2k_\zeta e^{-hk}.$$

Recall that for all $x_0 \in \Omega_r$ and for all $k \leq c_0 \log \frac{1}{r}$, $B(x_0, r) \cap g^{-k}(B(x_0, r)) = \emptyset$. We get

$$\begin{aligned}
& \mu_{B(x_0,2r)}\left(\tau_{B(x_0,2r)} \leq r^{-d_0}\right) \\
\leq & \sum_{k=0}^{r^{-d_0}} \mu_{B(x_0,2r)}\left(g^{-k}B(x_0,2r)\right) \\
= & \sum_{k=c_0 \log \frac{1}{2r}}^{\lfloor b_1 \log \frac{1}{2r} \rfloor} \mu_{B(x_0,2r)}\left(g^{-k}B(x_0,2r)\right) + \sum_{k=\lfloor b_1 \log \frac{1}{2r} \rfloor + 1}^{r^{-d_0}} \mu_{B(x_0,2r)}\left(g^{-k}B(x_0,2r)\right) \\
\leq & \sum_{k=c_0 \log \frac{1}{2r}}^{\lfloor b_1 \log \frac{1}{2r} \rfloor} 2k_\zeta e^{-hk} + \sum_{k=\lfloor b_1 \log \frac{1}{2r} \rfloor + 1}^{r^{-d_0}} \left(k_\zeta \mu(B(x_0,2r)) + 2k_\zeta e^{-hk}\right) \\
\leq & \sum_{k=c_0 \log \frac{1}{2r}}^{r^{-d_0}} 2k_\zeta e^{-hk} + \sum_{k=\lfloor b_1 \log \frac{1}{2r} \rfloor + 1}^{r^{-d_0}} k_\zeta \mu(B(x_0,2r)) \\
\leq & 2k_\zeta e^{-hc_0 \log \frac{1}{2r}} + r^{-d_0} k_\zeta \mu(B(x_0,2r)) \\
= & 2k_\zeta (2r)^{hc_0} + r^{-d_0} k_\zeta \mu(B(x_0,2r)).
\end{aligned}$$

This ends the proof. □

Proof of Proposition 3.13. By definition of $\underline{\varphi}$ and Lemma 3.14,

$$\begin{aligned}
\underline{\varphi}(a_0, \epsilon) &= \underline{\lim}_{r \to 0} \frac{1}{\log r} \log \mu\left(\Omega_r^c\right) \\
&\geq \underline{\lim}_{r \to 0} \frac{1}{\log r} \log r^{a_0} = a_0 > 0,
\end{aligned}$$

thus the proposition is proved. □

We are now able to prove Theorem 3.2

Proof of the Theorem 3.2. For $\gamma \in (0, 1)$, we get by Proposition 3.9 that

$$\underline{\psi}(\gamma\epsilon) = \Lambda^*(-d_\mu - \gamma\epsilon) > 0$$

and

$$\underline{\psi}(-\gamma\xi) = \Lambda^*(-d_\mu + \gamma\xi) > 0.$$

Moreover, by Proposition 3.13,

$$\underline{\varphi}(a_0, \epsilon) \geq a_0 > 0.$$

Thus, it follows form Theorem 2.4, that

$$\underline{f}(\epsilon) \geq \min\{(1-\gamma)\epsilon, \Lambda^*(-d_\mu - \gamma\epsilon)\} > 0 \tag{14}$$

and

$$\underline{f}(-\epsilon) \geq \min\left\{-\gamma(\epsilon + \xi) + a_0, \Lambda^*(-d_\mu - \gamma\epsilon), \Lambda^*(-d_\mu + \gamma\xi)\right\} > 0 \tag{15}$$

where $\Lambda^*(x) = -x + T^*(x)$ and the theorem is proved. □

4. Proof of the main result

In this section we prove the Theorem 2.4 using the method developed in [16]. We begin by the following lemma which will be needed in the proof of our main theorem.

Lemma 4.1. *Let* $(a_i(r))_{i=1,\dots,p}, a_i(r) > 0$. *If* $\gamma_i = \underline{\lim}_{r\to 0} \frac{1}{\log r} \log a_i(r) > 0$. *Then*

$$\underline{\lim}_{r\to 0} \frac{1}{\log r} \log \left(\sum_{i=1}^{p} a_i(r) \right) \geq \min_{i=1,\dots,p} \gamma_i.$$

Proof. For all $\epsilon > 0$ there exists $r_i > 0$ such that $r < r_i$ implies $a_i \leq r^{\gamma_i - \epsilon}$. Let $\epsilon > 0$ sufficiently small such that $\gamma_i - \epsilon > 0$. We have,

$$\sum_{i=1}^{p} a_i(r) \leq \sum_{i=1}^{p} r^{\gamma_i - \epsilon} \leq p r^{\min\{\gamma_i\} - \epsilon}$$

and this implies

$$\frac{1}{\log r} \left(\sum_{i=1}^{p} a_i(r) \right) \geq \min_{i=1,\dots,p} \{\gamma_i\} - \epsilon + \frac{\log p}{\log r}.$$

Finally,

$$\underline{\lim}_{r\to 0} \frac{1}{\log r} \log \left(\sum_{i=1}^{p} a_i(r) \right) \geq \min_{i=1,\dots,p} \{\gamma_i\} - \epsilon.$$

The result is proved since ϵ can be chosen arbitrarily small. □

Given $\epsilon, \xi > 0$, define

$$A_\epsilon(r) = \left\{ x \in X : \mu(B(x,r)) \geq r^{d_\mu + \epsilon} \right\} \tag{16}$$

and

$$A_{-\xi}(r) = \left\{ x \in X : \mu(B(x,r)) \leq r^{d_\mu - \xi} \right\}. \tag{17}$$

Proof of the Theorem 2.4. Let $\gamma \in (0,1)$. We have

$$\begin{aligned} \mu(\{x : \tau_r(x) \geq r^{-d_\mu - \epsilon}\}) &\leq \mu\left(\left\{ x \in A_{\gamma\epsilon}\left(\frac{r}{4}\right) : \tau_r(x) \geq r^{-d_\mu - \epsilon} \right\} \right) \\ &+ \mu\left(\left\{ x \in A^c_{\gamma\epsilon}\left(\frac{r}{4}\right) : \tau_r(x) \geq r^{-d_\mu - \epsilon} \right\} \right). \end{aligned}$$

Let us define the set

$$M_r = \left\{ x \in A_{\gamma\epsilon}\left(\frac{r}{4}\right) : \tau_r(x) \geq r^{-d_\mu - \epsilon} \right\}.$$

Let $\left(B\left(x_i, \frac{r}{2}\right)\right)_i$ be a family of balls of radius $r/2$ centered at points of $A_{\gamma\epsilon}(r)$ that covers M_r and such that $B\left(x_i, \frac{r}{4}\right) \cap B\left(x_j, \frac{r}{4}\right) = \emptyset \ if \ x_i \neq x_j$. We have

$$\begin{aligned} \mu\left(\left\{ x : \tau_r(x) \geq r^{-d_\mu - \epsilon} \right\} \right) &\leq \mu(\cup_i B_i \cap M_r) + \mu\left(\left\{ x \in A^c_{\gamma\epsilon}\left(\frac{r}{4}\right) : \tau_r(x) \geq r^{-d_\mu - \epsilon} \right\} \right) \\ &\leq \sum_i \mu(B_i \cap M_r) + \mu\left(A^c_{\gamma\epsilon}\left(\frac{r}{4}\right) \right). \end{aligned}$$

Using first the triangle inequality and then Kac's lemma and Markov inequality, we obtain

$$\mu(B_i \cap M_r) \le \mu\left(B_i \cap \left\{\tau_{B_i} \ge r^{-d_\mu-\epsilon}\right\}\right) \le r^{d_\mu+\epsilon}\int_{B_i}\tau_{B_i}d\mu = r^{d_\mu+\epsilon}.$$

Observe that $\sum_i \left(\frac{r}{4}\right)^{d_\mu+\gamma\epsilon} \le \sum_i \mu\left(B\left(x_i, \frac{r}{4}\right)\right) \le 1$. Thus, since the balls are disjoint it follows that the number of balls is bounded by $\left(\frac{1}{4}r\right)^{-d_\mu-\gamma\epsilon}$. Therefore,

$$\begin{aligned}\mu\left(\left\{x : \tau_r(x) \ge r^{-d_\mu-\epsilon}\right\}\right) &\le \sum_i \mu(B_i)r^{d_\mu+\epsilon} + \mu\left(A^c_{\gamma\epsilon}\left(\frac{r}{4}\right)\right)\\ &\le \left(\frac{1}{4}r\right)^{-d_\mu-\gamma\epsilon} r^{d_\mu+\epsilon} + \mu\left(A^c_{\gamma\epsilon}\left(\frac{r}{4}\right)\right)\\ &\le 4^{d_\mu+\gamma\epsilon}r^{(1-\gamma)\epsilon} + \mu\left(A^c_{\gamma\epsilon}\left(\frac{r}{4}\right)\right).\end{aligned}$$

Thus,

$$\frac{1}{\log r}\log\mu\left(\left\{x : \tau_r(x) \ge r^{-d\mu-\epsilon}\right\}\right) \ge \frac{1}{\log r}\log\left(4^{d_\mu+\gamma\epsilon}r^{(1-\gamma)\epsilon} + \mu\left(A^c_{\gamma\epsilon}\left(\frac{r}{4}\right)\right)\right).$$

Hence, by Lemma 4.1, we get

$$\begin{aligned}\underline{f}(\epsilon) &\ge \varliminf_{r\to 0}\frac{1}{\log r}\log\left(4^{d_\mu+\gamma\epsilon}r^{(1-\gamma)\epsilon} + \mu\left(A^c_{\gamma\epsilon}\left(\frac{r}{4}\right)\right)\right)\\ &\ge \min\left\{\varliminf_{r\to 0}\frac{1}{\log r}\log\left(4^{d_\mu+\gamma\epsilon}r^{(1-\gamma)\epsilon}\right), \varliminf_{r\to 0}\frac{1}{\log r}\log\mu\left(A^c_{\gamma\epsilon}\left(\frac{r}{4}\right)\right)\right\}\\ &= \min\left\{(1-\gamma)\epsilon, \underline{\psi}(\gamma\epsilon)\right\}.\end{aligned}$$

This proves the first statement.

Now, let us define

$$\Gamma_r = \left\{x \in A_{\gamma\epsilon}(2r) \cap A_{(-\gamma\xi)}(2r) : \tau_r(x) \le r^{-d_\mu+\epsilon}\right\}$$

and

$$D_r = \left\{x_0 : \mu_{B(x_0,2r)}(\tau_{B(x_0,2r)} \le r^{-d_\mu+\epsilon}) \le r^a\right\}.$$

Let $(B(x_i, 2r))_i$ be a family of balls of radius $2r$ centered at points of $A_{\gamma\epsilon}(2r) \cap D_r \cap A_{(-\gamma\xi)}(r)$ that covers $\Gamma_r \cap D_r$ and such that $B(x_i, r) \cap B(x_j, r) = \emptyset$ *if* $x_i \ne x_j$. We have

$$\begin{aligned}\mu\left(\left\{x : \tau_r(x) \le r^{-d_\mu+\epsilon}\right\}\right) &\le \mu\left(\left\{x \in A_{\gamma\epsilon}(2r) \cap D_r \cap A_{(-\gamma\xi)}(r) : \tau_r(x) \le r^{-d_\mu+\epsilon}\right\}\right)\\ &\quad +\mu\left(\left\{x \in (A_{\gamma\epsilon}(2r) \cap D_r \cap A_{(-\gamma\xi)}(r))^c : \tau_r(x) \le r^{-d_\mu+\epsilon}\right\}\right)\\ &\le \mu\left(\cup_i B(x_i, 2r) \cap \Gamma_r \cap D_r\right) + \mu\left(A^c_{\gamma\epsilon}(2r)\right) + \mu\left(D^c_r\right) + \mu\left(A^c_{(-\gamma\xi)}(r)\right).\end{aligned}$$

We remark that

$$\begin{aligned}\mu\left(\cup_i B(x_i, 2r) \cap \Gamma_r \cap D_r\right) &\le \sum_i \mu(B(x_i, 2r) \cap \Gamma_r \cap D_r)\\ &\le \sum_i \mu(B(x_i, 2r))\frac{1}{\mu(B(x_i, 2r))}\mu\left(B(x_i, 2r) \cap \left\{\tau_{B(x_i,2r)} \le r^{-d_\mu+\epsilon}\right\}\right)\end{aligned}$$

where the last inequality follows from $\left\{\tau_{B(x_i,r)} \leq r^{-d_\mu+\epsilon}\right\} \subset \left\{\tau_{B(x_i,2r)} \leq r^{-d_\mu+\epsilon}\right\}$. Therefore, by definition of D_r,

$$\begin{aligned} & \mu\left(\left\{x : \tau_r(x) \leq r^{-d_\mu+\epsilon}\right\}\right) \\ \leq & \sum_i \mu(B(x_i,2r))\mu_{(B(x_i,2r))}\left(\tau_{B(x_i,2r)} \leq r^{-d_\mu+\epsilon}\right) + \mu\left(A^c_{\gamma\epsilon}(2r)\right) + \mu\left(D^c_r\right) + \mu\left(A^c_{(-\gamma\xi)}(r)\right) \\ \leq & \sum_i \mu(B(x_i,2r))r^a + \mu\left(A^c_{\gamma\epsilon}(2r)\right) + \mu\left(D^c_r\right) + \mu\left(A^c_{(-\gamma\xi)}(r)\right). \end{aligned}$$

Observe that $\sum_i r^{d_\mu+\gamma\epsilon} \leq \sum_i \mu(x_i,r) \leq 1$. Thus, since balls are disjoint it follows that the number of balls is bounded by $r^{-d_\mu-\gamma\epsilon}$ and

$$\begin{aligned} \sum_i \mu(B(x_i,2r)) & \leq \sum_i (2r)^{d_\mu-\gamma\xi} \\ & \leq r^{-d_\mu-\gamma\epsilon}(2r)^{d_\mu-\gamma\xi} \\ & \leq 2^{d_\mu-\gamma\xi} r^{-\gamma(\epsilon+\xi)}. \end{aligned}$$

Then, we obtain that

$$\mu\left(\left\{x : \tau_r(x) \leq r^{-d_\mu+\epsilon}\right\}\right) \leq 2^{d_\mu-\gamma\xi} r^{-\gamma(\epsilon+\xi)+a} + \mu\left(A^c_{\gamma\epsilon}(r)\right) + \mu\left(D^c_r\right) + \mu\left(A^c_{(-\gamma\xi)}(r)\right).$$

Hence,

$$\underline{f}(-\epsilon) \geq \underline{\lim}_{r\to 0} \frac{1}{\log r} \log\left(2^{d_\mu-\gamma\xi} r^{-\gamma(\epsilon+\xi)+a} + \mu\left(A^c_{\gamma\epsilon}(r)\right) + \mu\left(D^c_r\right) + \mu\left(A^c_{(-\gamma\xi)}(r)\right)\right).$$

Finally, using the definitions of $\underline{\psi}$ and $\underline{\varphi}$ we get by Lemma 4.1 that

$$\underline{f}(-\epsilon) \geq \min\left\{-\gamma(\epsilon+\xi)+a, \underline{\psi}(\gamma\epsilon), \underline{\varphi}(a, d_\mu-\epsilon), \underline{\psi}(-\gamma\xi)\right\}.$$

This concludes the proof of the theorem. □

We finish with a brief proof of the Proposition 2.5.

Proof of the Proposition 2.5. Take $t = Cr^a, C > 0$. Making the first order expansion of e^{-t}, we have for $x \in \Omega_r$

$$\left|\mu_{B(x,r)}\left(\tau_{B(x,r)} > \frac{Cr^a}{\mu(B(x,r))}\right) - 1 + Cr^a + o(r^{2a})\right| \leq r^a,$$

which implies

$$\left|\mu_{B(x,r)}\left(\tau_{B(x,r)} < \frac{Cr^a}{\mu(B(x,r))}\right) + Cr^a + o(r^{2a})\right| \leq r^a.$$

So, it follows that

$$\mu_{B(x,r)}\left(\tau_{B(x,r)} < \frac{Cr^a}{\mu(B(x,r))}\right) < r^a.$$

Let N_r be a set defined by $N_r = \left\{x : \mu(B(x,r)) \geq r^{d_\mu+a-\epsilon}\right\}$. For $x \in N_r \cap \Omega_r$ we obtain

$$\mu_{B(x,r)}\left(\tau_{B(x,r)} < Cr^{-d_\mu+\epsilon}\right) < r^a.$$

Thus,

$$\mu\left(\left\{x : \mu_{B(x,2r)}\left(\tau_{B(x,2r)} \leq r^{-d_\mu+\epsilon}\right) > 2^a r^a\right\}\right) \leq \mu((N_{2r} \cap \Omega_{2r})^c) \leq \mu(N_{2r}^c) + \mu(\Omega_{2r}^c).$$

Finally, by Lemma 4.1, we get

$$\underline{\varphi}(a,\epsilon) \geq \min\{\underline{\psi}(a-\epsilon), b\}.$$

One can observe that the factor 2^a in the above definition of $\underline{\varphi}$ does not change the general result. □

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