Dimensions for recurrence times : topological and dynamical properties.

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Abstract

In this paper we give new properties of the dimension introduced by Afraimovich to characterize Poincaré recurrence and which we proposed to call Afraimovich-Pesin's (AP's) dimension. We will show in particular that AP's dimension is a topological invariant and that it often coincides with the asymptotic distribution of periodic points : deviations from this behavior could suggest that the AP's dimension is sensible to some "non-typical" points.

Contents

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1 Introduction

The Carathéodory-Pesin construction (see [14] for a complete presentation and historical accounts), has revealed to be a powerful unifying approach for the understanding of thermodynamical formalism and fractal properties of dynamically defined sets. A new application of this method has recently been proposed by Afraimovich [1] to characterize Poincaré recurrence : it basically consists in the construction of an Hausdorff-like outer measure (with the related transition point, or dimension), but with a few important differences. The classical Hausdorff measure (see for instance [7]) is constructed by covering a given set A with arbitrary subsets and by taking the diameter of these subsets at a power α to build up the Carathéodory sum. In Afraimovich's setting, the diameter is replaced with a decreasing function (gauge function) of the smallest first return time of the points of each set of the covering into the set itself. We proposed in [16] to call the related transition point, the Afraimovich-Pesin's (AP's) dimension of the set A.
The comparison with the usual Hausdorff measure reveals another difference : the choice of the type of sets in the coverings (arbitrary, open or closed sets) does not matter in the definition of the Hausdorff dimension, whereas it does in general for the AP's dimension (and indeed the use of arbitrary coverings would give us straightforward results). To understand these facts we introduce in Section 3 a large class of pre-measures and we study the behavior of the associated Borel measures : this will produce new properties in the Carathéodory-Pesin's setting.

The first interesting result is that for continuous dynamical systems on compact spaces X, the AP's dimension is preserved by homeomorphisms, it is thus a topological invariant. Moreover the AP's dimension of X coincides with the AP's dimension of the non-wandering set, when the map is restricted to it. These two properties are shared with the topological entropy and it may be asked whether it is possible to push forward this analogy. To this extent, the choice of the gauge function will play a fundamental role. The original paper of Afraimovich dealt with irrational rotations of the circle and for such systems the gauge function $1/t$ (where t is the smallest first return time of points of a set into itself) revealed to be a good choice to relate the AP's dimension to some Diophantine characteristics of the rotations number. Some heuristic (and very natural) arguments suggested us to introduce the gauge function e^{-t}, particularly adapted for dynamical systems with positive topological entropy.

With this gauge function, one can prove, as a first result, a general lower bound :

$$AP \geq \limsup_{k\to\infty} \frac{1}{k} \log \#\mathrm{Per}(k), \tag{1}$$

where $\#\mathrm{Per}(k)$ is the number of periodic points with smallest period k. In the case of subshifts of finite type with a finite alphabet, this bound becomes an equality, and tells us that the AP's dimension is equal to the topological entropy. The equality in (1) persists even when the alphabet is infinite, which is an expression, when the transition matrix is irreducible, of the Gurevič entropy.

At this point, two questions arise :

1. is the AP's dimension always equal to the topological entropy (for example on compact spaces) ?
2. is there always an equality in (1) with the gauge function e^{-t} ?

The answer to the first question is negative : we produce indeed a counterexample in Section 4.3. The second question is much more subtle and in fact unsolved. One way to attack it would be to exhibit an example of a minimal set with positive AP's dimension. Although there are (many) minimal sets with positive topological entropy, we are not able at the moment to show the same for the AP's dimension.

One step in the understanding of this problem is however done in Section 4.4. We introduce there the AP's dimension associated to an invariant measure, as the smallest AP's dimension over all subsets of full measure, in analogy with the Hausdorff dimension of the measure [20]. We first prove that the AP's dimension of the measure is metrically invariant. Then, when the measure is aperiodic and invariant, with the gauge function e^{-t}, the dimension is actually equal to zero.

The question therefore arises whether the periodic points are the only responsibles of the positiveness of the AP's dimension of a set or if other points contribute to it.

We conclude this introduction by addressing two more axes of investigation. Up to now, we worked with open (or closed) coverings in the construction of our dimension : we could alternatively use dynamical partitions (i.e. cylinders) and try to get another type of AP's dimension as the thermodynamic limit of some partition function. This direction has begun to be explored by Afraimovich in the spirit of a multifractal description of invariant sets [2, 3] ; instead we related such a thermodynamic limit to the large deviations of a local quantity which we called in [11] the "local rate of smallest return times for cylinders". Such a work is still in progress.

We finally point out that the original work of Afraimovich concerned minimal sets : in particular he used the AP's dimension, with the gauge function $1/t$, to classify irrational rotations. It would be interesting to push further this approach and use the AP's dimension as a topological invariant to classify systems with zero topological entropy. In these cases other invariant numbers exist, like the covering number and the complexity[5, 8, 12] , and they have been successfully applied to systems like rotations, exchange of intervals and substitutions. The computation of the AP's dimension for such systems and its comparison with the other invariants is an interesting and we think promising open field of research.

2 Preliminaries

The dynamical systems considered in this paper will be formed by a compact metric space X with distance d, the Borel $\sigma-$algebra Σ and a continuous application T on X; although several results remain true for discontinuous mappings (see remark 4.1 below). We finally consider invariant Borel (regular) probability measures μ.

The fundamental quantities investigated in this paper are Poincaré recurrences ; take first U a subset of X and define for each $x \in U$ the first return time into U as :

$$\tau_U(x) = \inf \left\{ k > 0 \;\middle|\; T^k x \in U \right\}.$$

We will use the convention that if the point x never returns to U then its return time is infinite.

For any invariant probability measure μ, Poincaré's recurrence theorem tells us that τ_U is finite for $\mu-$almost every point in each measurable subsets U. An improvement of this result, due to Kac [15], tells us that for any ergodic measure μ, we have the general relation about the mean value of τ_U over U with respect to μ :

$$\int_U \tau_U(x) \frac{\mathrm{d}\mu(x)}{\mu(U)} = \frac{1}{\mu(U)}. \tag{2}$$

(remark that we do not suppose that the transformation T is invertible.)

We want here to adopt another point of view : instead of looking at the mean return time, we are going to study the smallest possible return time into U.

The Poincaré recurrence of a *point* $\tau_U(x)$ as defined above leads us to define the first return time of a *set* : it is the infimum over all return times of the points of the set, and it can be written in three equivalent manners :

$$\tau(U) = \inf\{\tau_U(x) : x \in U\}, \tag{3}$$
$$\tau(U) = \inf\{k > 0 \ : T^k U \cap U \neq \emptyset\}, \tag{4}$$
$$\tau(U) = \inf\{k > 0 \ : T^{-k} U \cap U \neq \emptyset\}. \tag{5}$$

We call $\tau(U)$ the Poincaré recurrence of the set U. Note that (5) shows that $\tau(U) = \tau(T^{-1}U)$. It is also easy to verify that $\tau(U)$ is a monotonic function :

$$A \subset B \Rightarrow \tau(A) \geq \tau(B). \tag{6}$$

It is important to realize that in many cases the smallest return time of a set $\tau(U)$ and the return time of a point of the set $\tau_U(x)$ will be very different : for example, for cylinder sets of order n, the smallest return time is usually of order n, while the mean return time will grow, by Kac's lemma and Shannon-Mc Millan-Breiman Theorem, as e^{nh_μ}, where h_μ is the metric entropy of μ. We will give in the following sections, arguments and results to support such considerations.

As anticipated in the introduction, we will use the Poincaré recurrence of a set to find, via the Carathéodory construction over a subset $A \subset X$, a family of Borelian measures indexed by a parameter α. For each A, only one value of α will distinguish unambiguously between infinite and finite measures of A : we will show that this critical value of α, the AP's dimension, will bring a global information on A.

3 Carathéodory's construction

3.1 Construction of an outer measure

For any $A \subset X$, we define $\mathcal{R}^a_{\leq}(A, \epsilon)$ (resp. $\mathcal{R}^a_{=}(A, \epsilon)$) the collection of all *countable* coverings of A by subsets of X with diameter less or equal (resp. equal) to ϵ. The upper-script a stands to show that we use arbitrary sets in our covers.

In the same way, we denote by $\mathcal{R}^o_{\leq}$ (or $\mathcal{R}^o_{=}$) and $\mathcal{R}^f_{\leq}$ (or $\mathcal{R}^f_{=}$) the restrictions of the preceding collections to covers with respectively *open* and *closed* sets. In the case X is a one dimensional smooth manifold (usually compact subset of $\mathbb{R}$ or the circle), we will also consider covers by interval, denoted by $\mathcal{R}^i$.

We then define a set function, that we call *pre-measure*, $\Phi : 2^X \to \mathbb{R}^+$ with the property that $\Phi(\emptyset) = 0$. In the case Φ coincides with the diameter, we will set $\Phi \equiv \text{diam}$.

We then construct the Carathéodory sum

$$M^{\Phi}_{\alpha}(A, \epsilon) = \inf_{R \in \mathcal{R}_{\leq}(A,\epsilon)} \sum_{U \in R} \Phi(U)^{\alpha}. \tag{7}$$

(we do not precise here the type of sets we use in the covers ; otherwise we will write $M^{s,\Phi}_{\alpha}$ to significate that we use the collection of covers R^s.) It is easy to show that M^{Φ}_{α} is an *outer measure*.

Additionally, Φ might have some other properties :

$$\begin{aligned}
&\text{Property } (A): && \text{for any } U \subset V \Rightarrow \Phi(U) \leq \Phi(V) \quad \text{(monotonicity)}\ , \\
&\text{Property } (B): && \forall \epsilon > 0, \forall \text{ closed set } F \subset X, \\
& && \quad \exists O \text{ open such that } O \supset F, \Phi(O) - \Phi(F) < \epsilon, \\
&\text{Property } (C): && \forall U \subset X, \quad \Phi(\overline{U}) = \Phi(U),
\end{aligned}$$

(where $\overline{U}$ denotes the closure of U)

$$\text{Property } (D): \quad \forall \epsilon > 0, \exists \delta \text{ such that } |U| < \delta \Rightarrow \Phi(U) < \epsilon.$$

We now discuss how the choice of the type of covering influences the construction.

Theorem 3.1. *Let Φ be a pre-measure verifying Properties (A) and (B), then for any set A*

$$M_\alpha^{a,\Phi}(A,\epsilon) \leq M_\alpha^{o,\Phi}(A,\epsilon) \leq M_\alpha^{f,\Phi}(A,\epsilon), \tag{8}$$

and for any compact set K

$$M_\alpha^{o,\Phi}(K,\epsilon) = M_\alpha^{f,\Phi}(K,\epsilon). \tag{9}$$

Furthermore, if Φ has also property (C), then for any set A

$$M_\alpha^{a,\Phi}(A,\epsilon) = M_\alpha^{o,\Phi}(A,\epsilon) = M_\alpha^{f,\Phi}(A,\epsilon). \tag{10}$$

Proof. The first inequality in (8) is obvious. We now prove the second one : let $R^f = \{F_i, i = 1, 2, \dots\}$ be a countable cover of A by closed sets of diameter less than ϵ. Let $\delta > 0$, we construct a cover R^o by open sets in this way : for each F_i, we choose by property (B) an open set O_i containing F_i such that $\Phi(O_i) - \Phi(F_i) < \delta 2^{-i}$. By remarking that the collection of coverings R^o constructed in this way (let's call it $\mathcal{R}_1$) is a subset of $\mathcal{R}^o_\leq(A,\epsilon)$, we can write

$$\begin{aligned}
M^{f,\Phi}(A,\epsilon) + \delta = \inf_{R^f \in \mathcal{R}^f_\leq(A,\epsilon)} \sum_{i=1, F_i \in R^f}^{\infty} \Phi(F_i) + \delta &> \inf_{R^o \in \mathcal{R}_1} \sum_{i=1, O_i \in R^o}^{\infty} \Phi(O_i) \\
&\geq \inf_{R^o \in \mathcal{R}^o_\leq(A,\epsilon)} \sum_{O \in R^o} \Phi(O) = M^{o,\Phi}(A,\epsilon).
\end{aligned}$$

By taking δ arbitrary small, this proves (8).
Let now R^o be a countable cover of the compact set K with open sets of diameter less than ϵ. We denote with $\{O_i, i = 1, 2, \dots, p\}$ a finite subcover of K and put δ a Lebesgue number of this subcover. Then we construct a cover R^f with closed sets of the form

$$F_i = O_i \setminus \{x : d(x, X \setminus O_i) < \delta\}, \quad i = 1, 2, \dots, p.$$

We can verify that the collection of all the F_i is still a cover of A : for any point $x \in K$, there is a set $O_i \in R^o$ such that $d(x, X \setminus O_i) > \delta$. Thus, using the triangular inequality, we can affirm that $d(x, X \setminus F_i) > 0$, and so $x \in F_i$. Since the collection of all covers R^f constructed in this way (we call it $\mathcal{R}_2$) is a subset of $\mathcal{R}^f_{\leq}(K, \epsilon)$, and using the monotonicity of Φ, we have

$$M^{o,\Phi}(K, \epsilon) \geq \inf_{R^f \in \mathcal{R}_2} \sum_{i=1}^{\#R^f} \Phi(F_i) \geq M^{f,\Phi}(K, \epsilon). \tag{11}$$

This shows (9).
In order to prove (10), we just need to prove $M^{f,\Phi}_\alpha(A, \epsilon) \leq M^{a,\Phi}_\alpha(A, \epsilon)$. For each countable cover $R^a = \{A_i, i = 1, 2, \dots\}$ of A by arbitrary sets, we construct a closed cover $R^f = \{F_i \equiv \overline{A_i}\}$ of A. The collection of covers R^f constructed in this way (we call it $\mathcal{R}_3$) is a subset of $\mathcal{R}^f(A, \epsilon)$, so we have

$$M^{f,\Phi}_\alpha(A, \epsilon) \leq \inf_{R \in \mathcal{R}_3} \sum_{V \in R} \Phi(V) = M^{a,\Phi}_\alpha(A, \epsilon).$$

□

We can find a similar result as equation (10) in [17], but only for the case of Hausdorff measures, which is covered by our theorem.

3.2 Construction of a Borelian measure ; dimensions

We now take the limit $\epsilon \to 0$, which exists because $M^\Phi_\alpha(A, \epsilon)$ increases when ϵ goes to zero:

$$m^\Phi_\alpha(A) = \lim_{\epsilon \to 0} M^\Phi_\alpha(A, \epsilon). \tag{12}$$

The set function m^Φ_α is a *borelian measure* (as shown for example in [7]).

From now on we will work exclusively with Borel subset of X.

Note that all the properties stated for the outer measure $M^{s,\Phi}_\alpha$ in Theorem 3.1 remain true for the corresponding Borel measure $m^{s,\Phi}_\alpha$. In particular one can improve (9) by showing that

$$m^{o,\Phi}_\alpha(A) = m^{f,\Phi}_\alpha(A)$$

holds for any Borel set A whenever $m^{o,\Phi}_\alpha(A)$ and $m^{f,\Phi}_\alpha(A)$ are *finite* (it is just an easy consequence of the regularity of Borel measures on metric spaces and the inner approximation by compact sets). We do not know whether this result is true in general : one should prove the analogous of Theorem 48, p.97 in Rogers[17] for Hausdorff measures, which in our case would become : for any $\lambda > 0$ and any Borel set A with $m^{s,\Phi}_\alpha(A) > \lambda$ there exists a compact set $K \subset A$ such that $m^{s,\Phi}_\alpha(K) > \lambda$ (which is trivial to prove if the measures are $\sigma-$finite).

We now come back to the pre-measures : if Φ has property (D), i.e. it goes uniformly to zero when $|U|$ goes to zero, then we meet all the conditions of Carathéodory's construction, as described by Pesin in [14]. It is well known, then, that there exists a unique non-negative number $\alpha_c(A)$ such that

$$m_\alpha(A) = \begin{cases} \infty & \text{if } 0 < \alpha < \alpha_c(A) \\ 0 & \text{if } \alpha > \alpha_c(A) \end{cases} \tag{13}$$

In the cases we are interested in, property (D) might not be verified by the pre-measure, which put us out of the usual construction. However, we still can define a *critical exponent* (or dimension) : namely, we define the critical exponent of a set $A \subset X$ as

$$\alpha_c(A) = \sup\{\alpha > 0 : m_\alpha^\Phi(A) = \infty\}. \tag{14}$$

It is always well defined and non-negative if we adopt the convention that $\sup \emptyset = 0$; moreover it is monotone, that is $A \subset B \Rightarrow \alpha_c(A) \leq \alpha_c(B)$. Remark that this exponent may not be "net", that is to say that it could happen that $m_\alpha(A)$ is non zero for $\alpha > \alpha_c$.

We now state a Lemma which will be useful in the next section :

Lemma 3.2. *Let X_n be a countable sequence of subsets of X, such that each of them has a net critical exponent $\alpha_c(X_n)$ as defined in* (13)*, then*

$$\alpha_c\Big(\bigcup_n X_n\Big) = \sup_n\{\alpha_c(X_n)\},$$

and furthermore $\alpha_c(\bigcup_n X_n)$ is a net critical exponent.

Proof. Let $\alpha > \sup\{\alpha_c(X_n)\}$, then $m_\alpha(\bigcup_n X_n) \leq \sum_n m_\alpha(X_n) = 0$. Conversely, suppose $\alpha < \sup\{\alpha_c(X_n)\}$, then $\exists n$ such that $m_\alpha(X_n) = \infty$, so $m_\alpha(\bigcup_n X_n) = \infty$. □

We now show that, whenever the pre-measure Φ is invariant under conjugations by uniform homeomorphisms, then the measure m_α^Φ and dimension α_c are also invariant under the same conjugations.

Proposition 3.1. *Let (X, d) and (X', d') two metric spaces (not necessarily compact). Let $h : X \to X'$ a uniform homeomorphism. Suppose that we have the pre-measures $\Phi : X \to \mathbb{R}^+$ and $\Phi' : X' \to \mathbb{R}^+$ such that $\forall A \subset X, \Phi(A) = \Phi'(h(A))$. Then the measures m_α^Φ and $m_\alpha^{\Phi'}$, constructed on X and X' both with open covers, closed covers or covers by interval (in the one dimensional case only), are equal for any α. Thus the dimensions are also equal.*

Proof. For simplicity, we suppose that we are using open covers, the proof remains exactly the same with other type of covers (closed or by interval). We write the uniform continuity

$$\forall \delta > 0, \exists \epsilon(\delta) \text{ which goes to zero when } \delta \to 0 \text{ such that } \forall x, y \in X, \text{ if } d(x,y) < \epsilon(\delta) \text{ then } d'(h(x), h(y)) < \delta.$$

Let $A \subset X$ and $A' \subset X'$ such that $h(A) = A'$. Let $\mathcal{R}'_{\leq}(A', \delta)$ be the set of all covering γ' of A' with open sets of diameter less than δ. Let $h(\mathcal{R}_{\leq}(A, \epsilon(\delta)))$ be the set of all transformed coverings of A with open sets of diameter less than $\epsilon(\delta)$, namely $h(R) \equiv \{h(U), U \in R\}$. Then $\mathcal{R}'_{\leq}(A', \delta)$ contains $h(\mathcal{R}_{\leq}(A, \epsilon(\delta)))$. This shows that

$$M_\alpha^\Phi(A, \epsilon(\delta)) = \inf_{R \in \mathcal{R}_{\leq}(A, \epsilon(\delta))} \sum_{U \in R} \Phi(U)^\alpha \geq \inf_{R \in \mathcal{R}'_{\leq}(A', \delta)} \sum_{U \in R} \Phi'(U)^\alpha = M'^{\Phi'}_\alpha(A', \delta).$$

Then,

$$m_\alpha^\Phi(A) = \lim_{\delta \to 0} M_\alpha^\Phi(A, \epsilon(\delta)) \geq m'^{\Phi'}_\alpha(A').$$

Now, by reversing A and A''s rules, one can apply the same idea to obtain the opposite inequality, which yields

$$m_\alpha^\Phi(A) = m'^{\Phi'}_\alpha(A').$$

It is then obvious that $\alpha_c(A) = \alpha'_c(A')$. □

3.3 Dimension associated to a measure

In analogy with the Hausdorff dimension of the measure [20], we now introduce the critical exponent for any Borel probability measure μ. There are two standard ways to do that, precisely we set :

$$\alpha_c(\mu) \equiv \inf \left\{ \alpha_c(A) \;\middle|\; A \subset X, \mu(A) = 1 \right\}, \tag{15}$$

and we may also put :

$$\alpha'_c(\mu) = \sup_{\delta > 0} \inf \left\{ \alpha_c(A) \;\middle|\; A \subset X, \mu(A) > 1 - \delta \right\}. \tag{16}$$

These two definitions often coincide, more precisely we have :

Lemma 3.3. *Suppose that $\Phi(B_\epsilon(x))$ is measurable with respect to μ on X and that it goes to zero when ϵ goes to zero for $\mu-$almost every x in X, then*

$$\alpha_c(\mu) = \alpha'_c(\mu).$$

Proof. Replace ϵ with a decreasing sequence ϵ_m which goes to zero for $m \to \infty$. Let

$$a_n = \inf \left\{ \alpha_c(A) \;\middle|\; A \subset X, \mu(A) > 1 - \frac{1}{n} \right\}.$$

Then we choose a countable sequence X_n of sets with measure $\mu(X_n) > 1 - \frac{1}{n}$ and such that $0 < \alpha_c(X_n) - a_n < \frac{1}{n}$. Then we apply Egorov's theorem to each X_n by finding a measurable subset $X'_n \subset X_n$ still of measure $\mu(X'_n) > 1 - \frac{1}{n}$, on which the pre-measure has property (D) (in fact now $\Phi(B_{\epsilon_m}(x))$ is uniformly convergent over X'_n, thus $\forall \delta > 0$, one can find m_δ independent of $x \in X'_n$ such that $U \subset X'_n$ of diameter less than ϵ_{m_δ} is contained in a ball of radius ϵ_{m_δ} and such that $\Phi(U) < \Phi(B_{\epsilon_{m_\delta}}(x)) < \delta$). Then, since $\bigcup_n X'_n$ has full measure, using Lemma 3.2, we have $\alpha'_c(\mu) = \sup_n \alpha_c(X'_n) = \alpha_c(\bigcup_n X'_n) \geq \alpha_c(\mu)$. The opposite equality is trivial. □

We now prove that the critical exponent of a measure is a metric invariant.

Proposition 3.2. *Suppose the pre-measure Φ satisfies the hypothesis of Lemma 3.3 and is invariant under conjugations by measurable isomorphisms. Then the critical exponent of the measure μ is invariant under the same conjugations.*

Proof. Let Ψ denotes the isomorphism mod 0 between the two measurable spaces (X, Σ, μ) and (Y, Σ', ν). Define a_n as in the proof of the previous Lemma, and choose a sequence of countable subsets X_n with measure $\mu(X_n) > 1 - \frac{1}{n}$ and verifying $0 < \alpha_c(X_n) - a_n < \frac{1}{n}$. By applying Lusin's Theorem on each of the X_n we can find a compact set $\hat{X}_n \subset X_n$ still of measure $\mu(\hat{X}_n) > 1 - \frac{1}{n}$ on which Ψ becomes an homeomorphism. By Proposition 3.1 we then have

$$\alpha_c(\hat{X}_n) = \alpha_c\left(\Psi(\hat{X}_n)\right)$$

and then $\alpha_c(\mu) = \sup_n a_n = \sup_n \alpha_c(\hat{X}_n) = \sup_n \alpha_c(\Psi(\hat{X}_n)) \geq \alpha_c(\nu)$ since $\nu(\Psi(\hat{X}_n)) = \mu(\hat{X}_n) > 1 - \frac{1}{n}$. By interchanging the role of the two spaces we finally get the desired equality. □

3.4 Capacities

We now return to covers by sets which have all the same diameter ϵ ; by making the same construction as before we get another set function

$$R^\Phi_\alpha(A, \epsilon) = \inf_{R \in \mathcal{R}_=(A,\epsilon)} \sum_{U \in R} \Phi(U)^\alpha. \tag{17}$$

(where the absence of upper-script in $\mathcal{R}_=$ means that we do not precise the type of covering.)

The limit in ϵ does not in general exist, so we distinguish the lim sup and lim inf and write them $\overline{r}_\alpha^\Phi$ and $\underline{r}_\alpha^\Phi$. From these, we get two critical exponents, which are called in general capacities, and we note them $\overline{\text{cap}}$ and $\underline{\text{cap}}$. We have the obvious inequalities

$$\dim(A) \leq \underline{\text{cap}}(A) \leq \overline{\text{cap}}(A) \tag{18}$$

for any subset $A \subset X$.

We now state a sufficient condition for the equality between dimension and lower capacity.

Lemma 3.4. *Let A a Borelian subset of X. Suppose there exists d_H such that the Hausdorff measure $m_{d_H}^{diam}(A)$ is non zero and that $\overline{r}_{d_H}^{diam}(A)$ is finite (under these assumptions, the number d_H is the Hausdorff dimension and capacity of the set A). Suppose that we have a pre-measure verifying $\frac{1}{C}g(|U|) \leq \Phi(U) \leq Cg(|U|)$ where g is an increasing function of the diameter with $g(0) = 0$ and C a constant independent of U. Then*

$$\alpha_c^\Phi(A) = \underline{\text{cap}}^\Phi(A). \tag{19}$$

Proof. Since $m_{\mathrm{d}_H}^{\text{diam}}(A) > 0$, there exists a constant C' such that, for any small enough $\epsilon > 0$, for any open countable cover $\{U_i\} \in \mathcal{R}_{\leq}(A, \epsilon)$, we have $\sum_i a_i^{\mathrm{d}_H} > C'$ where $a_i \equiv |U_i|$. Then for any of these covers we have

$$\begin{aligned} \sum_i \Phi(U)^\alpha &\geq \frac{1}{C} \sum_i \frac{a_i^{\mathrm{d}_H} g(a_i)^\alpha}{a_i^{\mathrm{d}_H}} \geq \frac{C'}{C} \inf_i \left\{ \frac{g(a_i)^\alpha}{a_i^{\mathrm{d}_H}} \right\} \\ &\geq \frac{C'}{C} \inf_{\epsilon > \epsilon' > 0} \{N_{\epsilon'}(A) g(\epsilon')\} \geq \frac{C'}{C^2} \inf_{\epsilon > \epsilon' > 0} \{R(A, \epsilon')\}. \end{aligned}$$

(Where $N_\epsilon(A)$ denotes the smallest number of open ball of diameter ϵ necessarily to cover the set A.)
Thus, $M_\alpha^\Phi(A, \epsilon) \geq \frac{C'}{C^2} \inf_{\epsilon > \epsilon' > 0}\{R_\alpha^\Phi(A, \epsilon')\}$ and therefore $m_\alpha^\Phi(A) \geq \frac{C'}{C^2} \underline{r}^\Phi(A)$. Using (18), we get the result. □

4 Definition of Afraimovich-Pesin's dimension

We now apply our construction to the study of Poincaré's recurrence. Afraimovich-Pesin's dimension will be defined as the critical exponent with a particular pre-measure depending on the set function τ previously defined.

Let $h : \mathbb{N} \to \mathbb{R}^+$ be a non-increasing function, taking finite and non zero values, such that $h(k) \xrightarrow[k \to \infty]{} 0$. Then we take $\Phi(U) = h(\tau(U))$.

Lemma 4.1. *Suppose that X is a compact space, then the pre-measure $\Phi(U) = h(\tau(U))$ has the properties (A) and (B).*

Proof. Property (A) follows at once from the monotonicity property of τ (see (6)), and from the fact that h is a non-increasing function. We now prove property (B) : let U be a closed subset of X and $\epsilon > 0$. We consider two cases :

1. $\tau(U) = k < \infty$: for $j = 1, \dots, k-1$, one has $T^j U \cap U = \emptyset$, and since U is compact, and so is $T^j U$, one has $\min_{j=1,\dots,k-1} d(T^j U, U) = d > 0$. Then, by uniform continuity of T, one can find an open set V containing U such that $\forall j, \forall x \in T^j V, d(x, T^j U) < \frac{d}{2}$. Thus we can affirm that $T^j V \cap V = \emptyset, j = 1, \dots, k-1$, which shows that $\tau(U) = \tau(V)$, and so $\Phi(V) - \Phi(U) = 0 < \epsilon$.

2. $\tau(U) = \infty$: in this case, $\Phi(U) = 0$. We choose an integer k such that $h(k) < \epsilon$, then we can apply the same argument as in the first case : we can find an open set V containing U with $\tau(V) \geq k$, and so $\Phi(V) - \Phi(U) < \epsilon$.

□

We are now in the position to apply Theorem 3.1, and for that we will consider pre-measures which are function of the Poincaré recurrence as defined in the beginning of this section. The critical exponent will be called either dimension for Poincaré recurrence, or Afraimovich-Pesin's dimension.

In order to fully define AP's dimension, we need to precise two more things : the exact choice of the function Φ and the choice of the type of covering.

As anticipated in the introduction, the first function used by Afraimovich was $h(t) = \frac{1}{t}$ in the context of rotations. In this paper we concentrate on the study of dynamical systems using the function $h(t) = e^{-t}$, which seems particularly adapted for those with positive topological entropy.

We now analyze in detail the influence of the type of covering.

4.1 The choice of the sets used in the cover

We suggested three possibilities : arbitrary covers, open covers and closed covers. In the one dimensional case, covers by intervals are also relevant [1]. We now show that the first one gives straightforward results :

Proposition 4.1. *The outer measure $M_{\alpha}^{a,\Phi}$ constructed, with a pre-measure as defined in the beginning of this section and using arbitrary covers, is concentrated on the periodic points. Furthermore, with a pre-measure $\Phi(U) = e^{-\tau(U)}$, and if the number of periodic points of smallest period k is finite for every k, then*

$$\alpha_c^{\Phi}(X) = \overline{\lim}_{k\to\infty} \frac{1}{k} \log \#\mathrm{Per}(k),$$

where $\#\mathrm{Per}(k)$ denotes the number of periodic points with smallest period k. If, on the contrary, $\exists k$ such that $\#\mathrm{Per}(k) = +\infty$, then $\alpha_c^{\Phi}(X) = +\infty$

[1]since the property "a set is an interval" is a topological property, and so is invariant under topological conjugacy.

Proof. We define an equivalence relation for the points of X : for any $x, y \in X$, we will say that they are equivalent if there is $k_x > 0$ and $k_y > 0$ for which $T^{k_x}x = T^{k_y}y$. We call orbits the equivalence classes. We now define a slice of an orbit : let x be a point of X, a slice of an orbit is a set $S(x) = \bigcup_{k>0} T^{-k}T^k x$. It is obviously a subset of an orbit. In the case where T is invertible, a slice of an orbit is simply one point of the orbit, whereas in the non-invertible case, it corresponds to those points of the orbit which correspond to the same position in time. Note that one can reconstruct the whole orbit from a slice simply by iterating it with T and T^{-1}, which one could not do in general from only one point of the orbit (if T is not invertible).

Then we construct a set U by taking, for each orbit, one (and no more) point x and then its slice $S(x)$ (we need to use the Axiom of Choice to do that). If F denotes the set of all periodic points, one can check that $\tau(U \setminus F) = \infty$, but even more : $\forall k \in \mathbb{N}, \tau(T^k U \setminus F) = \infty$. Thus, the countable family of sets $U_k \equiv T^k U \setminus F$ is a countable cover of $X \setminus F$ whose members have all infinite Poincaré recurrence. We can construct covers whose members have diameter as small as we want by cutting in a countable number of pieces each set U_k. (This can be done by taking their intersection with members of a cover by ball of diameter ϵ for example.) This shows that $M_\alpha^{a,\Phi}(X \setminus F)$=0.

We now prove the second statement of the proposition. With the pre-measure $\Phi(U) = e^{-\tau(U)}$, each periodic point of period k has a measure $M_\alpha^{a,\Phi}(x) = e^{-k}$, thus $M_\alpha^{a,\Phi}(F) = \sum_{k>0} \#\mathrm{Per}(k)e^{-\alpha k}$. We see in particular that the critical exponent cannot be net. If $\alpha_c(F)$ is infinite, then the result follows immediately. Otherwise, let $\alpha > \alpha_c(F)$ and $\epsilon > 0$; in this case, for k big enough we have $\#\mathrm{Per}(k)e^{-\alpha k} < \epsilon$, which immediately gives $\overline{\lim}_{n\to\infty} \frac{1}{n} \log(\#\mathrm{Per}(n)) \leq \alpha$. This still holds if we replace α with $\alpha_c(F)$. For the lower bound, let now $\alpha < \alpha_c(F)$; then the sum is infinite which implies that $\forall \delta > 0$ we can find a infinite number of integers k such that $\#\mathrm{Per}(k)e^{-\alpha k} > e^{-\delta k}$, thus $\overline{\lim}_{n\to\infty} \frac{1}{n} \log(\#\mathrm{Per}(n)) \geq \alpha - \delta$. The result then follows since δ is arbitrary, and by sending α to $\alpha_c(F)$. □

We now state an easy but useful result

Lemma 4.2. *Periodic points are the only atoms of the measure m_α^Φ constructed with open or closed covers.*

Proof. Since a point is a compact set, one just has to prove the theorem for closed covers. We take a non-periodic point x, then we consider the cover consisting of the point itself ; since it is not periodic, it's return time is infinite. Thus $m_\alpha^\Phi(\{x\}) = 0$. Let's suppose now that x is a periodic point. Then $\tau(\{x\})$ is equal to the period of x, which is finite. Thus $m_\alpha^\Phi(\{x\}) = h(\tau(\{x\}))$ is finite and non zero. □

Remark 4.1. *We note that by using open and closed covers, Theorem 3.1 applies by virtue of Lemma 4.1 and in principle we could get different values for the AP's* *dimension but with an important exception : if the application T is continuous, the AP's dimensions will be the same on compact sets and even more, in the situation discussed in Section 3.2. This will permit to get an interesting property of the AP's dimension for the non-wandering set (see Theorem 4.5).*

However for any application T, an easy consequence of the preceding Proposition is that a general bound in terms of periodic points for the AP's dimension constructed with open or closed covers can be stated (point 4 of Theorem 4.5).

4.2 Some properties of Afraimovich-Pesin's dimension

The first important property, is that this dimension is a topologically invariant number. We recall that, for example, the Hausdorff dimension depends in general on the metric (actually, it is invariant only under Lipschitz conjugation [7]).

Theorem 4.3. *For any $\alpha > 0$, the Borelian measure m_α^Φ and the dimension α_c^Φ, constructed with open or closed covers and in the one dimensional case with open or closed interval covers, are invariant under topological conjugations by uniform homeomorphisms.*

Corollary 4.4. *If T is invertible, then for any $\alpha \geq 0$, m_α^Φ is an invariant measure.*

Proof of the Theorem. It is just a consequence of Proposition 3.1. □

Proof of the Corollary. One just has to use the theorem, setting $h = T$. □

The next theorem establishes other properties of AP's dimension, the most important being the one that says that AP's dimension over X coincides with AP's dimension restricted to the set of non-wandering points, which is exactly what happens for the topological entropy.

Theorem 4.5. *Dimension for Poincaré recurrence has the following properties (using open or closed covers) :*

1. *if we use the pre-measure $\Phi(U) = \mathrm{e}^{-\tau(U)}$, then for any $k > 0$, we have $\alpha_c^\Phi(T^k, X) \leq k\alpha_c^\Phi(T, X)$,*
2. *if we use the pre-measure $\Phi(U) = \frac{1}{\tau(U)}$, then for any $k > 0$, we have $\alpha_c^\Phi(T^k, X) \leq \alpha_c^\Phi(T, X)$,*
3. *$\alpha_c^\Phi(T, X) = \alpha_c^\Phi(T, NW) = \alpha_c^\Phi(T_{|NW}, NW)$, where NW denotes the set of non-wandering points.*

4. *if we use the pre-measure* $\Phi(U) = \mathrm{e}^{-\tau(U)}$*, and if the number of periodic points of smallest period* k *is finite for every* k*, then there is the lower-bound*

$$\alpha_c^{\Phi}(X) \geq \overline{\lim}_{k\to\infty} \frac{1}{k} \log \#\mathrm{Per}(k),$$

where $\#\mathrm{Per}(k)$ *denotes the number of periodic points with smallest period* k*. If, on the contrary,* $\exists k$ *such that* $\#\mathrm{Per}(k) = +\infty$*, then* $\alpha_c^{\Phi}(X) = +\infty$ [2].

Remark. *There are examples of diffeomorphisms of the unit disk where strict inequality holds in the two first points of this theorem (see Section 4.3). However, these constructions depend heavily on the combinatorics of the periodic points, and these maps are somewhat unnatural. That is why we might think that for a large class of dynamical systems the equality holds. We recall that for topological entropy, the following equality holds :*

$$h_{\mathrm{top}}(T^k) = k h_{\mathrm{top}}(T).$$

The last point gives a lower-bound to AP's dimension with the periodic points. We recall that there exists a similar lower-bound for topological entropy for expansive maps (see, for example, [19] p.178) :

$$h_{\mathrm{top}} \geq \overline{\lim}_{k\to\infty} \frac{1}{k} \log \#\mathrm{Fix}(k),$$

where $\#\mathrm{Fix}(k)$ *denotes the number of fixed points of* T^k*.*

Proof. 1. First, let us remark that for any subset $U \subset X$,

$$k\tau(T^k, U) \geq \tau(T, U),$$

where $\tau(T,U)$ denotes the Poincaré recurrence for the application T, with respect to U. (There is equality when $\tau(T,U)$ is a multiple of k.) Therefore, one can write

$$\begin{aligned} M_\alpha(T^k, A, \epsilon) &\equiv \inf_{\gamma\in\mathcal{R}_{\leq}(A,\epsilon)} \left\{ \sum_{U\in\gamma} e^{\alpha\tau(T^k,U)} \right\} \\ &\leq \inf_{\gamma\in\mathcal{R}_{\leq}(A,\epsilon)} \left\{ \sum_{U\in\gamma} e^{\frac{\alpha}{k}\tau(T,U)} \right\} = M_{\alpha/k}(T, A, \epsilon). \end{aligned}$$

By taking the limit $\epsilon \to 0$ we obtain

$$m_\alpha(T^k, A) \leq m_{\alpha/k}(T, A),$$

and

$$\alpha_c(T^k, A) \leq k\alpha_c(T, A).$$

[2]This shows in particular that AP's dimension of the identity map is infinite.

2. similar proof than for the previous one, but the factor k disappears because of the different choice of the pre-measure.

3. Let us denote W the complementary of NW, i.e. the set of all wandering point. We recall that a point x is non-wandering if, for all open sets U containing x, there is t such that $T^t U \cap U \neq \emptyset$. A wandering point then is simply a point contained in an open set U that has infinite Poincaré recurrence (i.e. $\tau(U) = \infty$). Let $\epsilon(x) > 0$ be the radius of a ball centered on x that has infinite Poincaré recurrence. It is defined for any wandering point x. Let $R = \{B(x, \epsilon(x)), x \text{ wandering point}\}$ be a cover of W. Since X is separable, and hence is W, we can get from this cover a countable cover R_c. We can then use this cover as a particular cover in $M_\alpha^{o,\Phi}(W, \epsilon)$ (provided we chose $\epsilon(x)$ small enough, which we can always do), which yields

$$M_\alpha^{o,\Phi}(W, \epsilon) = 0.$$

Then obviously

$$m_\alpha^{o,\Phi}(X) = m_\alpha^{o,\Phi}(NW) + m_\alpha^{o,\Phi}(W) = m_\alpha^{o,\Phi}(NW).$$

Now, since X is compact, so is NW, which implies by (9) that these equalities hold also with closed covers. Then, since NW is a closed set, we can restrict the family of closed covers to those completely contained in NW, which proves the last equality of the proposition.

4. this is an easy application of the second part of Theorem 4.1.

□

4.3 An example with $\alpha_c(T^k, X) < k\alpha_c(T, X)$

We give the example with $\Phi(U) = \mathrm{e}^{-\tau(U)}$. The idea relies on a special combinatorics of periodic points. We construct a discrete subset of $\mathbb{R}^2$: $X = \{x_k^l : k \text{ odd}, l = 1, \dots, G_k(2^k)\}$, (the function $G_k(n)$ gives the smallest number multiple of k and greater than n), and we put the points on concentric circles of radius $\frac{1}{k}$, with angle proportional to l. The reason why we put the points in such a way is that we want to show an example on a compact set (here, X is compact). We can now define the dynamics : $T(x_k^l) = x_k^{l+G_k(2^k)/k \mod G_k(2^k)}$. Obviously, any point x_k^l has period k, and there are exactly $G_k(2^k)$ points with odd period k, and none with even period.

One can check that $\alpha_c(T, X) = \overline{\lim}_{k\to\infty} \frac{1}{k} \log \#\mathrm{Per}(T, k) = \log 2$. Now, because all periods are odd, they will not be changed if we use the dynamics T^2 instead of T. So, AP's dimension remains the same for T and T^2.

To make the example a *little bit* less artificial, one can turn it into a diffeomorphism of the unit disk by defining a new dynamic on the unit disk such that all points fall on the previously defined periodic points. Then, X is the set of non-wandering points of the dynamic on the disk, and by Theorem 4.5 part 3, the previous result remains.

This shows by the way that AP's dimension and topological entropy do not necessarily coincide (the latter is actually equal to zero here).

4.4 AP's dimension associated to a measure

We now return to the definition of the critical exponent associated to a measure as in Section 3.3.

Lemma 4.6. *Suppose that T is a measurable endomorphism preserving an aperiodic probability measure μ. Let $A \subset X$, then for any $N \in \mathbb{N}$ and any $\epsilon > 0$ one can find N pairwise disjoint sets $U^k \subset A, k = 0, 1, \ldots, N-1$ with Poincaré recurrence $\tau(U^k) \geq N$, and such that $\mu(\bigcup_{k=0}^{N-1} U^k) > \mu(A) - \epsilon$.*

Proof. Let $N \in \mathbb{N}$ and $\epsilon > 0$. By Rokhlin-Halmos' theorem [3] [6], we can find a set U such that $T^{-k}U$ for $k = 0, 1, \ldots, N-1$ are pairwise disjoint, and furthermore such that $\mu(\bigcup_{k=0}^{N-1} T^{-k}U) > 1 - \epsilon$. Obviously $\tau(U) \geq N$. Moreover, $\tau(T^{-k}U) = \tau(U)$ from the definition (5). We now define $U^k = A \cap T^{-k}U$ and these sets have all the desired properties. □

Remark that we can choose the sets U^k compact because of the regularity of the measure.

Theorem 4.7. *Suppose that (X, T, μ) satisfies the same conditions as in Lemma 4.6 and that the box dimension of the space X is finite. Then with a pre-measure $\Phi(U) = \mathrm{e}^{-\tau(U)}$, the AP's dimension associated to an aperiodic invariant probability measure μ according with definition* (16) *and constructed with arbitrary or closed covers, is equal to* 0.

Proof. We make use of Lemma 4.6 : Let $\delta > 0$, we take some compact sets $U_n^k, k = 0, 1, \ldots, n-1$ with Poincaré recurrences $\tau(U_n^k) \geq n$ and such that $\mu(\bigcup_{k=0}^{n-1} U_n^k) > 1 - \frac{\delta}{2^n}$. We write $X_n = \bigcup_{k=0}^{n-1} U_n^k$. Then we define $\{U_{n+l}^k\}$ in the same way, but now with $A = X_{n+l-1}$, so that the resulting X_{n+l} is a subset of X_{n+l-1} and its measure is such that $\mu(X_{n+l}) \geq \mu(X_{n+l-1}) - \frac{\delta}{2^{n+l}}$. We define in this way a monotone sequence of compact sets X_{n+l} with measure $\mu(X_{n+l}) > 1 - \sum_{l'=0}^{l} \frac{\delta}{2^{n+l'}}$. We take the intersection of these sets $V = \bigcap_{l=0}^{\infty} X_{n+l}$ and get a compact set with measure $\mu(V) > 1 - \delta$. For each l, the collection $\{U_{n+l}^k\}_k$ is a cover of V by $n + l$ closed sets for which Poincaré recurrence is at least equal to $n + l$. We can cut these sets in order to obtain a cover with closed sets of

[3] Usually the Rokhlin-Halmos' theorem is stated for invertible systems. The generalization to the non-invertible case can be done using the natural extension; a more direct approach due to E.Lesigne [unpublished] has been communicated to us by Y.Lacroix.

diameter less than $\frac{1}{n+l}$. It is possible for l large enough to get a cover with less than $(n+l)^{1+d}$ members, where d is a number greater than the box dimension of the space (which is supposed to be finite). Thus, we have the upper-bound $M_\alpha^{f,\Phi}(V, \frac{1}{n+l}) \leq (n+l)^{1+d} e^{-\alpha(n+l)}$, and for any $\alpha > 0$,

$$m_\alpha^{f,\Phi}(V) = 0.$$

Since the measure of V can be arbitrarily close to 1, the dimension of the measure, following the second definition (16), is 0. This proves the theorem when we cover with closed sets, and this remains true with arbitrary covers. □

Remark 4.2. *The last result can be strongly improved if we make the additional assumption that T is continuous. In this case, inequality* (3.1) *in Lemma 3.3 and Theorem 8 apply, hence the two dimensions associated to the measure are equal to zero, using arbitrary, closed or* open *covers.*

5 Examples

We now present some systems for which AP's dimension can be calculated.

5.1 Systems with pre-measure $\Phi(U) = e^{-\tau(U)}$: AP's dimension for subshifts of finite type

We will work on the space $\Omega = \{0, \dots, p-1\}^{\mathbb{N}}$ of all semi-infinite sequences $\omega = \omega_1\omega_2\dots$, with the product topology. The alphabet might be infinite ($p = \infty$). We consider the shift to the left σ such that $\sigma(\omega_1\omega_2\omega_3\dots) = \omega_2\omega_3\dots$. Subshifts of finite type are restrictions of the shift on some invariant subsets of Ω. See [13] for a complete description of these systems. Many dynamical systems are topologically conjugate to subshifts of finite type [10, 4].

We define $n-$cylinders : $C_{\alpha_1,\dots,\alpha_n} = \left\{\omega \in \Omega \;\middle|\; \omega_1 = \alpha_1, \dots, \omega_n = \alpha_n\right\}$.

Theorem 5.1. *For subshifts of finite type, with finite or infinite alphabet, such that* $\#\mathrm{Per}(k)$ *is finite for every k, we have*

$$\alpha_c(\Omega) = \overline{\lim}_{k\to\infty} \frac{1}{k} \log \#\mathrm{Per}(k). \tag{20}$$

When the alphabet is finite, this tells us that AP's dimension is equal to topological entropy. For infinite alphabet, when the subshift is irreducible, this limit corresponds to the "loop entropy", or Gurevič entropy [9].

Proof. If the return time of a cylinder is k, then it is easy to check that it contains a periodic point with period k. We consider a particular cover of X : the one with $n-$cylinders. Then, in this cover, the number of cylinders with first return time k is at most the number of periodic points with period k. Thus

$$M_\alpha^\Phi(X, 2^{-n}) \leq \sum_k \#\mathrm{Per}(k)\mathrm{e}^{-\alpha k}.$$

(we use a metric such that the diameter of an $n-$cylinder is 2^{-n}.)

In the limit $\epsilon \to 0$, this upper-bound remains true, and using the lower bound of Theorem 4.5 part 4, we get the result. □

5.2 Systems with pre-measure $\Phi(U) = \frac{1}{\tau(U)}$

When V.Afraimovich has introduced his AP's dimension, he first studied it on minimal sets [1]. One of the reason is that in this case, he could prove that the pre-measure has property (D), which implies that, as in the usual Carathéodory construction, we have a net critical exponent (see (13)).

We can give a general lower bound for AP's dimension with pre-measure $\Phi(U) = \frac{1}{\tau(U)}$. The idea is to use Kac's lemma.

Theorem 5.2. *Let (X, T) be a minimal dynamical system. Let A a subset of X. If there exists a Borelian ergodic measure μ such that $\mu(A) > 0$, then AP's dimension of A, with open covers and pre-measure $\Phi(U) = \frac{1}{\tau(U)}$, is such that*

$$\alpha_c(A) \geq 1.$$

This particularly implies that

$$\alpha_c(X) \geq 1.$$

Proof of the Theorem. Using Kac's Lemma, we can say that for any set U, we have $\tau(U) \leq \int_U \tau_U(x) \frac{\mathrm{d}\mu(x)}{\mu(U)} = \frac{1}{\mu(U)}$. For any cover R_ϵ of A with open sets of diameter less than ϵ, we can write

$$\sum_{U \in R_\epsilon} \frac{1}{\tau(U)} \geq \sum_{U \in R_\epsilon} \mu(U) \geq \mu(A) > 0.$$

This shows that $m_1(A) > 0$. Since X is minimal, there exists α_c such that

$$m_\alpha(A) = \begin{cases} \infty & \text{if } \alpha < \alpha_c \\ 0 & \text{if } \alpha > \alpha_c \end{cases}$$

This proves that $\alpha_c \geq 1$. □

We now recall Afraimovich's result in [1], on which we can apply our last theorem. It concerns irrational rotations of the circle $\mathcal{S}^1$. Let ω an irrational number, then we define the dynamic $T_\omega : \mathcal{S}^1 \to \mathcal{S}^1$ such that $T_\omega(x) = x + \omega \mod 1$. Every orbit is dense in the space and thus it is a minimal dynamical system.

We define for any irrational number, ω the Diophantine characteristic $\nu(\omega)$ which is the supremum over all ν for which there exists an infinity of pairs of relatively prime numbers p, q such that

$$\left|\omega - \frac{p}{q}\right| < \frac{1}{q^{\nu+1}}.$$

It is well known that $\nu(\omega) \geq 1$ [18], it can also become infinite.

Theorem 5.3 ([1]). *Taking the pre-measure $\Phi(U) = \frac{1}{\tau(U)}$ and using covers by open interval, irrational rotation of the circle with a rotation number ω has AP's capacities*

$$\overline{\mathrm{cap}}(\mathcal{S}^1) = \nu(\omega).$$

Therefore, we see that, when $\nu(\omega) = 1$ (for example when ω is *badly approximable* as defined in [18]), by using Theorem 5.2 :

$$\alpha_c(\mathcal{S}^1) = \underline{\mathrm{cap}}(\mathcal{S}^1) = \overline{\mathrm{cap}}(\mathcal{S}^1) = 1.$$

An interesting question is what we would get if we consider the AP's dimension constructed with open covers rather than interval covers. This question is especially important in multi-dimensional rotations where we cannot anymore consider covers by intervals. Covering by balls is possible [1], but it would not give us an invariant number ; this is the reason why we prefer to cover with open sets.

6 Concluding remarks

We presented in this paper some new characterizations of the Afraimovich-Pesin's dimension, and we addressed a certain number of open questions. We recall at least two of them :
(i) the possible coincidence of the AP's dimension with asymptotic growth of number of periodic points even when we cover with open or closed sets. A possible counterexample would show the influence of some "non-generic" points on the AP's dimension.
(ii) the role of the AP's dimension to classify systems with zero topological entropy when we use the pre-measure $\Phi(U) = \frac{1}{\tau(U)}$.

Moreover it would be interesting to compute the AP's dimension for more general classes of systems, including non-continuous applications, and to have algorithms to evaluate the AP's dimension numerically ; we did it for the logistic map in [16].

As anticipated in the introduction, it would be also interesting to get (a sort of) AP's dimension as the thermodynamic limit of some partition function : this could open the path to a multifractal analysis of such a dimension.

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