EXPONENTIAL LAW FOR RANDOM SUBSHIFTS OF FINITE TYPE

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Abstract. In this paper we study the distribution of hitting times for a class of random dynamical systems. We prove that for invariant measures with super-polynomial decay of correlations hitting times to dynamically defined cylinders satisfy exponential distribution. Similar results are obtained for random expanding maps. We emphasize that what we establish is a quenched exponential law for hitting times.

1. Introduction

The theory of random dynamical systems has been introduced to obtain a better modelisation of motions of particles or physical phenomena in general. Indeed, instead of iterating the same transformation one can add some random noise or small perturbations, or more generally work with a family of transformation randomly chosen to represent the errors of approximations or observations. One can see the review [20] for an introduction to this theory.

Another theory which has been widely studied in the last few years (e.g. the review [27]) is the quantitative description of recurrence in deterministic dynamical systems. More precisely, let (X, T, μ) be a measure preserving dynamical system, the hitting time of a point $x \in X$ to a set A is defined by

$$\tau_A(x) := \inf\{k > 0, T^k x \in A\},$$

when $x \in A$, we will speak of return time. This theory is interested in the behaviour of $\tau_A(x)$ when $\mu(A) \to 0$.

A first point of view is to study the return time of a point x in its r-neighborhood (i.e. $A = B(x, r)$) and its behavior when $r \to 0$. It has been proved [6, 4, 26] that for rapidly mixing systems $\tau_{B(x,r)}(x) \underset{r\to 0}{\sim} r^{-d_\mu(x)}$ where $d_\mu(x)$ is the pointwise dimension of the measure μ in x. We refer the reader to [12, 13] for the same type of results for hitting time and [25, 24] for generalizations.

Another point of view is to study the distribution of return times and hitting time statistics (we can cite the review of Coelho [7] and Abadi and Galves [1] and also the article of Collet, Galves and Schmitt [9] which is one of the first results on this domain). More precisely, we define the distribution of normalized hitting time by

$$F_A^{hit}(t) = \mu\left(\left\{x \in X : \tau_A(x) > \frac{t}{\mu(A)}\right\}\right)$$

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and the distribution of normalized return times by

$$F_A^{ret}(t) = \frac{1}{\mu(A)}\mu\left(\left\{x \in A : \tau_A(x) > \frac{t}{\mu(A)}\right\}\right).$$

These works studied the convergence in law of the distribution of normalized hitting and return times when $\mu(A) \to 0$ for sets A well-chosen (for example cylinders of a partition).

Haydn, Lacroix and Vaienti [15] proved that the limit of the distribution of the return times exists if and only if the limit of the distribution of the hitting times exists. Moreover, an exponential distribution was proved for various families of dynamical systems: Axiom A diffeomorphisms [16], Markov chains [23], some rational transformations [14], uniformly expanding transformations of the interval [8], and some non-uniformly hyperbolic systems [17, 28]. Recently, Freitas, Freitas and Todd [10, 11] linked hitting time statistics to extreme value theory.

Despite the fact that the quantitative study of Poincaré recurrence has been widely studied, the quantitative approach to recurrence for random dynamical systems remains much incomplete. A first attempt was obtained recently by Marie and Rousseau [21] where they study the random recurrence rate for super-polynomially mixing random dynamical systems. More precisely, they proved that for rapidly mixing systems, the quenched recurrence rates are equal to the pointwise dimensions of the stationary measure. One can also see the recent article of Aytaç, Freitas and Vaienti [2] on law of rare events for random dynamical systems.

In this paper, we prove, in Section 3 and 4, an exponential law for the distribution of the hitting time for random subshifts of finite type assuming some rapid decay of correlations while similar results are proved in Section 5 for some random expanding maps. Our main theorems are stated precisely in Section 2 and we apply our result to some random subshift and random expanding maps in Section 6.

2. Statement of the main results

We first give the definition of a random subshift of finite type. Let $(\Omega, \theta, \mathbb{P})$ be an invertible ergodic measure preserving system, set $X = \mathbb{N}^{\mathbb{N}}$ and let $\sigma : X \to X$ denote the shift. Let $b : \Omega \to \mathbb{N}$ be a random variable such that $\mathbb{E}(\log b) < \infty$. Let $A = \{A(\omega) = (a_{ij}(\omega)) : \omega \in \Omega\}$ be a random transition matrix, i.e. for any $\omega \in \Omega$, $A(\omega)$ is a $b(\omega) \times b(\theta\omega)$-matrix with entries in $\{0, 1\}$, at least one non-zero entry in each row and each column and such that $\omega \mapsto a_{ij}(\omega)$ is measurable for any $i \in \mathbb{N}$ and $j \in \mathbb{N}$. For any $\omega \in \Omega$ define the subset of the integers $X_\omega = \{1, \ldots, b(\omega)\}$ and

$$\mathcal{E}_\omega = \{\underline{x} = (x_0, x_1, \ldots) \colon x_i \in X_{\theta^i\omega} \text{ and } a_{x_i x_{i+1}}(\theta^i\omega) = 1 \text{ for all } i \in \mathbb{N}\} \subset X,$$

$$\mathcal{E} = \{(\omega, \underline{x}) \colon \omega \in \Omega, \underline{x} \in \mathcal{E}_\omega\} \subset \Omega \times X.$$

We consider the random dynamical system coded by the skew-product $S : \mathcal{E} \to \mathcal{E}$ given by $S(\omega, \underline{x}) = (\theta\omega, \sigma\underline{x})$. Let ν be an S-invariant probability measure with marginal $\mathbb{P}$ on Ω and let $(\mu_\omega)_\omega$ denote its decomposition on $\mathcal{E}_\omega$, that is, $d\nu(\omega, \underline{x}) = d\mu_\omega(\underline{x})d\mathbb{P}(\omega)$. The measures μ_ω are called the *sample measures*. We denote by $\mu = \int \mu_\omega \, d\mathbb{P}$ the marginal of ν on X.

For $\underline{y} \in X$ we denote by $C^n(\underline{y}) = \{\underline{z} \in X : y_i = z_i \text{ for all } 0 \le i \le n-1\}$ the *n-cylinder* that contains $\underline{y}$. Set $\mathcal{F}_0^n$ as the sigma-algebra in X generated by all the n-cylinders.

Our hypothesis on b guarantees that the metric entropy $h_\nu(S, \Omega \times \mathcal{F}_0^1)$ is finite and we will denote it by h.

We assume the following: there are constants $h \geq h_0 > 0$, $c > 0$, a random variable $C \in L^p(\Omega, \mathbb{P})$ for some $p \in (0,1]$, a constant $q > \frac{h}{h_0}(1 + \frac{3}{p})$ and a function $\alpha(g)$ satisfying $\alpha(g)g^q \to 0$ when $g \to +\infty$ such that for all m, n, $A \in \mathcal{F}_0^n$ and $B \in \mathcal{F}_0^m$:

(I) (polynomial decay of correlations) the marginal measure μ satisfies

$$\left|\mu(A \cap \sigma^{-g-n} B) - \mu(A)\mu(B)\right| \leq C_0 \alpha(g);$$

(II) (exponential small cylinders) $\mu_\omega(C^n(\underline{y})) \leq c e^{-h_0 n}$ for any $\underline{y} \in X$ and $n \geq 1$, for $\mathbb{P}$-almost every $\omega \in \Omega$;

(III) (fibered polynomial decay of correlations)

$$\left|\mu_\omega(A \cap \sigma^{-g-n} B) - \mu_\omega(A)\mu_{\theta^{n+g}\omega}(B)\right| \leq C(\omega)\alpha(g)$$

for $\mathbb{P}$-almost every $\omega \in \Omega$.

Given $A \subset X$ consider the hitting time $R(\underline{x}, A) = \inf\{k \geq 1 \colon \sigma^k \underline{x} \in A\}$ and set $R_n(\underline{x}, \underline{y}) = R(\underline{x}, C^n(\underline{y}))$ for $\underline{x}, \underline{y} \in X$.

Theorem 1. *We assume that hypothesis (I), (II) and (III) hold. For μ-almost every $\underline{y}$, $\mathbb{P}$-almost every ω and all $t \geq 0$ we have*

$$\mu_\omega\left(\underline{x} \in X \colon R_n(\underline{x}, \underline{y}) > \frac{t}{\mu(C^n(\underline{y}))}\right) \to e^{-t}, \quad \text{as } n \to \infty. \tag{1}$$

This can be view as a *quenched* exponential law for hitting time. We provide some applications in Section 6, while a similar result is obtained for random endomorphisms with some rapidly mixing conditions in Section 5.

The later convergence together with integration over Ω and dominated convergence theorem yields the following *annealed* version:

Corollary 2. *Under the same hypothesis, for μ-almost every $\underline{y}$ and $t \geq 0$,*

$$\mu\left(\underline{x} \in X \colon R_n(\underline{x}, \underline{y}) > \frac{t}{\mu(C^n(\underline{y}))}\right) \to e^{-t}, \quad \text{as } n \to \infty.$$

It is natural to conjecture that the convergence in distribution in the theorem holds almost everywhere with respect to the measure ν, that is:

Conjecture 3. *For ν-a.e. $(\omega, \underline{y})$ the convergence* (1) *in the theorem holds.*

Despite the strong similarity of the conjecture with the theorem, these two statements are not comparable. In particular, the corollary would not follow from the conjecture, since even in the simple case of a random Bernoulli measure the sample measures μ_ω and μ could well be mutually singular; see Example 19 for details.

We now provide a similar result for a class of maps satisfying some decay of correlations reminiscent of expanding maps. Let $(\Omega, \theta, \mathbb{P})$ be an invertible ergodic measure preserving transformation, $X_\omega \subset X$ be subsets of a compact metric space X, let $f_\omega : X_\omega \to X_{\theta(\omega)}$ be a bimeasurable map and consider the associated random dynamical system described by the skew-product $S : \mathcal{E} \to \mathcal{E}$ given by $S(\omega, x) = (\theta(\omega), f_\omega(x))$. As before, let ν be an S-invariant probability measure with marginal $\mathbb{P}$ on Ω, let $(\mu_\omega)_\omega$ denote its decomposition and let $\mu = \int \mu_\omega \, d\mathbb{P}$ be the marginal of ν on X.

Replace (I) by (I'): There exists $\gamma(\ell)$ going to zero faster than any power of ℓ such that: for φ Lipschitz on X and ψ measurable bounded on X

$$\left|\int_\Omega \int_X \varphi\,(\psi \circ f_\omega^\ell) d\mu_\omega d\mathbb{P}(\omega) - \int_X \varphi d\mu \int_X \psi d\mu\right| \leq C_0 \gamma(\ell) Lip(\varphi) \sup|\psi|.$$

Replace assumption (II) by (II') : it exists $d_0 > 0$ such that $\underline{d}_\mu(x) > d_0$ ν-almost everywhere and $\mu_\omega(B(x,r)) \leq cr^{d_0}$ for all x, r, ω.

Replace assumption (III) by (III'): for φ Lipschitz on X and ψ measurable bounded on X

$$\left|\int_X \varphi\,(\psi \circ f_\omega^\ell)\, d\mu_\omega - \int_X \varphi\, d\mu_\omega \int_X \psi\, d\mu_{\theta^\ell \omega}\right| \leq C(\omega)\gamma(\ell) Lip(\varphi) \sup|\psi|.$$

Include assumption (IV'): there are constants $a, b > 0$ such that for all x and $r, \rho > 0$ it holds $\mu(B(x, r+\rho)) \leq \mu(B(x,r)) + r^{-b}\rho^a$.

Include assumption (V'): the system is random-aperiodic, i.e.

$$\nu\{(\omega, x) \in \mathcal{E} : \exists n \in \mathbb{N},\, f_\omega^n x = x\} = 0.$$

Theorem 4. *If the random dynamical system satisfies (I')-(V') then for μ-a.e. $y \in X$, there exists random variables Δ_r defined on Ω such that $\Delta_r \to 0$ in probability and*

$$\sup_{t \geq 0} \left|\mu_\omega\left(x \in X : \tau^\omega_{B(y,r)}(x) > t/\mu(B(y,r))\right) - e^{-t}\right| \leq \Delta_r(\omega).$$

Remark 5. *The method does give the convergence almost surely in $\omega \in \Omega$ as in the previous section. We recall, however, that the convergence in probability of Δ_r implies that a.s. ω there exists a sequence $r_n^\omega \to 0$ such that $\Delta_{r_n^\omega}(\omega) \to 0$ as $n \to \infty$.*

The question of the speed of convergence could be aborded in some situations. A quite interesting question is also to understand if the presence of exponential law for return times implies that the fluctuations of repetition times and empirical entropies do coincide.

3. Estimates for general random systems and sets

In this section we describe general results that will be used in the proofs of our main results, and whose strategy follows the line of [17]. They are valid for any random dynamical system f_ω acting on X, where θ preserves the probability $\mathbb{P}$ on Ω. We write $\tau_A^\omega(x)$ the first k such that $f_\omega^k(x) \in A$. Here $A \subset X$ is a measurable deterministic set and $f_\omega^k = f_{\theta^{k-1}(\omega)} \circ \cdots \circ f_{\theta(\omega)} \circ f_\omega$. Consider

$$\delta_\omega(A) = \sup_{j \geq 1} \left|\mu_\omega(\tau_A^\omega(\cdot) > j)\, \mu_\omega(A) - \mu_\omega(A \cap \{\tau_A^\omega(\cdot) > j\})\right|.$$

Since θ is invertible, by σ-invariance of ν and almost everywhere uniqueness of the decomposition $\nu = \int \mu_\omega\, d\mathbb{P}(\omega)$ we get that the set

$$\Omega' = \{\omega \in \Omega \colon \forall i,\ (f_\omega^i)_* \mu_\omega = \mu_{\theta^i \omega}\}$$

has full $\mathbb{P}$-probability.

Lemma 6. *For all $\omega \in \Omega'$, integer k and measurable $A \subset X$ we have*

$$\left|\mu_\omega(\tau_A^\omega(\cdot) > k) - \prod_{i=1}^k (1 - \mu_{\theta^i \omega}(A))\right| \leq \sum_{i=1}^k \delta_{\theta^i \omega}(A) \prod_{j=1}^{i-1} (1 - \mu_{\theta^j \omega}(A)).$$

Proof. For any integer $i \geq 1$ we have

$$\begin{aligned}\mu_\omega(\tau_A^\omega(\cdot) > i+1) &= \mu_\omega(f_\omega^{-1}(A^c \cap \{\tau_A^{\theta\omega}(\cdot) > i\})) \\ &= \mu_{\theta\omega}(\tau_A^{\theta\omega}(\cdot) > i) - \mu_{\theta\omega}(A \cap \{\tau_A^{\theta\omega}(\cdot) > i\}).\end{aligned}$$

Therefore $\left|\mu_\omega(\tau_A^\omega(\cdot) > i+1) - (1-\mu_{\theta\omega}(A))\,\mu_{\theta\omega}(\tau_A^{\theta\omega}(\cdot) > i)\right| \leq \delta_{\theta\omega}(A)$. An immediate recursive substitution argument finishes the proof of the lemma. □

The proof of Theorem 1 is based on the previous lemma. The strategy is to prove that the term $\prod_{i=1}^k (1-\mu_{\theta^i\omega}(A))$ is almost surely convergent to e^{-t}, and that the error term in the right hand side goes to zero almost surely.

Since the exponential distribution is continuous, the convergence (1) for any $t \geq 0$ is equivalent to the convergence for a countable dense set of t's. Henceforth, to establish the theorems it is sufficient to show that for any $t > 0$ we have the convergence $\mathbb{P}$-almost surely.

Let then $t > 0$ be fixed. Given $A \subset X$ let $k = k_{A,t} = \lfloor t/\mu(A) \rfloor$ and define

$$M_{A,t}(\omega) = \sum_{i=1}^{k_{A,t}} \mu_{\theta^i\omega}(A).$$

Lemma 7. *We have the equivalence as* $\sup_\omega \mu_\omega(A) \to 0$

$$\prod_{i=1}^{k_{A,t}} (1-\mu_{\theta^i\omega}(A)) \sim e^{-M_{A,t}(\omega)}.$$

Proof. This result is a consequence of the following simple and instrumental result: if $0 < \varepsilon \leq 1/2$ and $x_1, \ldots, x_k \in [0, \varepsilon]$ then

$$\exp\left(-(1+2\varepsilon)\sum_{i=1}^k x_i\right) \leq \prod_{i=1}^k (1-x_i) \leq \exp\left(-(1-2\varepsilon)\sum_{i=1}^k x_i\right).$$

This finishes the proof. □

Observe that by stationarity the expectation of $M_{A,t}$ is

$$\mathbb{E}(M_{A,t}) = \int_\Omega \sum_{i=1}^{k_{A,t}} \mu_{\theta^i\omega}(A)\, d\mathbb{P}(\omega) = k_{A,t}\mu(A), \tag{2}$$

which by definition of $k_{A,t}$ already shows that $\mathbb{E}(M_{A,t}) \to t$ as $\mu(A) \to 0$.

Next, the error term $\sum_{i=1}^k \delta_{\theta^i\omega}(A)$ in Lemma 6 decomposes as a mixing term and short entrance or return time terms as follows. Let $g \leq k$ be an integer and set

$$G_{A,k,g}(\omega) = \sum_{i=1}^k \mu_{\theta^i\omega}(A \cap \{\tau_A^{\theta^i\omega}(\cdot) \leq g\}),$$

$$H_{A,k,g}(\omega) = \sum_{i=1}^k \sup_{j \geq 1} \left|\mu_{\theta^i\omega}(A \cap (f_\omega^g)^{-1}\{\tau_A^{\theta^{i+g}}\omega(\cdot) > j\}) - \mu_{\theta^i\omega}(A)\mu_{\theta^{i+g}\omega}(\tau_A^{\theta^{i+g}}\omega(\cdot) > j)\right|,$$

$$K_{A,k,g}(\omega) = \sum_{i=1}^k \mu_{\theta^i\omega}(A)\,\mu_{\theta^i\omega}(\tau_A^{\theta^i\omega}(\cdot) \leq g).$$

The gap g allows to exploit the mixing assumptions, related to $H_{A,k,g}$, provided that the probabilities of hitting or returning into A before time g, related to $G_{A,k,g}$ and $K_{A,k,g}$, are small since the whole error term is estimated as follows.

Lemma 8. *For all $\omega \in \Omega'$, any measurable set $A \subset X$ and any integers $g \leq k$ we have* $\sum_{i=1}^{k} \delta_{\theta^i\omega}(A) \leq G_{A,k,g}(\omega) + H_{A,k,g}(\omega) + K_{A,k,g}(\omega)$.

Proof. We have

$$\begin{aligned}\delta_\omega(A) &= \sup_{j\geq 1} |\mu_\omega(\tau_A^\omega(\cdot) > j)\mu_\omega(A) - \mu_\omega(A \cap \{\tau_A^\omega(\cdot) > j\})| \\ &\leq \mu_\omega(\tau_A^\omega(\cdot) \leq g)\, \mu_\omega(A) + \mu_\omega(A \cap \{\tau_A^\omega(\cdot) \leq g\}) \\ &\quad + \sup_{j \geq g} \left| \mu_{\theta^g\omega}(\tau_A^{\theta^g\omega}(\cdot) > j-g)\, \mu_\omega(A) - \mu_\omega(A \cap (f_\omega^g)^{-1}\{\tau_A^{\theta^g\omega}(\cdot) > j-g\}) \right|.\end{aligned}$$

Thus the lemma follows by summing up the the previous terms along the finite piece of orbit of ω by θ. $\square$

To summarize, to prove that the limiting law is a.s. exponential we are led to prove that $M_{A,t} \to t$ and G_A, H_A and K_A goes to zero as A shrinks to a typical reference point.

4. Proofs for the random subshifts

In this section we will prove Theorem 1 and the proof now follows the line of [27]. Consider the set

$$X' = \left\{ \underline{y} \in X \colon \limsup_{n\to\infty} -\frac{1}{n} \log \mu(C^n(\underline{y})) \leq h \right\}.$$

We already noticed that our hypothesis guarantee that the metric entropy $h_\nu(S, \Omega \times \mathcal{F}_0^1)$ is finite. Therefore, by Shannon-McMillan-Breiman theorem (see [29]) we obtain that $\limsup_{n\to\infty} -\frac{1}{n} \log \mu_\omega(C^n(\underline{y})) = h$ for ν-almost every $(\omega, \underline{x})$. Thus it follows from the Jensen's inequality that $\mu(X') = \nu(\Omega \times X') = 1$.

We fix some $t > 0$ and take $A = C_n(\underline{y})$. For simplicity we denote $M_{A,t}$ by M_n and $k_{A,t}$ by k_n. We forget also the dependence on $\underline{y}$, g and t for the other random variables introduced in the previous section and hence we write G_n, H_n, K_n for notational simplicity.

Lemma 9. *For all $\underline{y} \in X'$ we have $M_n \to t$, $\mathbb{P}$-almost surely.*

Proof. Let

$$\Omega_n = \left\{ \omega \in \Omega \colon \sum_{i=1}^{k_n} C(\theta^i\omega) \leq k_n^{\frac{3}{p}} \right\}.$$

We estimate the second moment of M_n on the set Ω_n

$$\mathbb{E}(M_n^2 1_{\Omega_n}) = \sum_{i,j=1}^{k_n} \int_{\Omega_n} \mu_{\theta^i\omega}(A)\mu_{\theta^j\omega}(A) d\mathbb{P}(\omega).$$

Let $\varepsilon > 0$ and consider now $m = m_n = \lfloor e^{h_0 n/(1+\varepsilon)} \rfloor$. Near the diagonal, that is when $|i-j| < m$, using hypothesis (II) we have that

$$\begin{aligned}\sum_{|i-i|<m} \int_{\Omega_n} \mu_{\theta^i\omega}(A)\mu_{\theta^j\omega}(A)d\mathbb{P}(\omega) &\le \sum_{|i-j|<m} ce^{-h_0 n}\int_{\Omega} \mu_{\theta^i\omega}(A)d\mathbb{P}(\omega)\\ &\le 2cme^{-h_0 n}k_n\mu(A)\\ &\le 2ctme^{-h_0 n}.\end{aligned}$$

Far from the diagonal, the independence hypotheses (I) and (III) yield

$$\begin{aligned}\sum_{|i-j|\ge m} \int_{\Omega_n} \mu_{\theta^i\omega}(A)\mu_{\theta^j\omega}(A)d\mathbb{P}(\omega) &\le 2\sum_{j\ge i+m}\int_{\Omega_n} C(\theta^i\omega)\alpha(m-n)d\mathbb{P}(\omega)+\\ &\qquad + \int_{\Omega}\mu_{\theta^i\omega}(A\cap\sigma^{-(j-i)}A)d\mathbb{P}(\omega)\\ &\le 2k_n k_n^{\frac{3}{p}}\alpha(m-n) + 2\sum_{j\ge i+m}\mu(A\cap\sigma^{-(j-i)}A)\\ &\le 2k_n^{1+\frac{3}{p}}\alpha(m-n) + k_n^2\mu(A)^2 + k_n^2 C_0\alpha(m-n).\end{aligned}$$

On the other hand, since the random variable $C \in L^p(\Omega,\mathbb{R})$ for some $p \in (0,1]$ it follows by Markov inequality that

$$\mathbb{P}(\Omega_n^c) = \mathbb{P}\left(\left(\sum_{i=1}^{k_n} C(\theta^i\omega)\right)^p > k_n^3\right) \le \mathbb{P}\left(\sum_{i=1}^{k_n} C(\theta^i\omega)^p > k_n^3\right) \le k_n^{-2}\,\mathbb{E}(C^p).$$

Thus, we simply have

$$\mathbb{E}(M_n^2 1_{\Omega_n^c}) \le \mathbb{P}(\Omega_n^c)\sup_{\Omega} M_n^2 \le k_n^{-2}\mathbb{E}(C^p)k_n^2 ce^{-h_0 n}.$$

Combining these estimates with (2) which gives $\mathbb{E}(M_n) = k_n\mu(A)$ we finally get a control on the variance of M_n

$$\begin{aligned}\operatorname{var} M_n &= \mathbb{E}(M_n^2 1_{\Omega_n}) + \mathbb{E}(M_n^2 1_{\Omega_n^c}) - \mathbb{E}(M_n)^2\\ &\le 2ctme^{-h_0 n} + 2k_n^{1+\frac{3}{p}}\alpha(m-n) + C_0 k_n^2\alpha(m-n) + \mathbb{E}(C^p)ce^{-h_0 n}.\end{aligned}$$

Thus, one can choose ε small enough such that $\sum_n \operatorname{var} M_n < \infty$. Indeed, for n large enough $k_n^2\alpha(m-n) < k_n^{1+\frac{3}{p}}\alpha(m-n) < 2^q t^{1+\frac{3}{p}} e^{n\gamma}$ where $\gamma = (h+\varepsilon)(1+\frac{3}{p}) - q\frac{h_0}{1+\varepsilon}$ and by definition of q, $\gamma < 0$ if ε is sufficiently small. It is a classical result that any sequence of centered random variables (X_n) with $\sum_n \operatorname{var} X_n < \infty$ is such that $X_n \to 0$ a.s. (since $\sum_n X_n$ is almost everywhere convergent). Hence, $M_n - \mathbb{E}(M_n) \to 0$ a.s., from which the conclusion follows since $\mathbb{E}(M_n) \to t$. □

We now prove that all random variables used in Lemma 8 converge to zero as n tends to infinity. We fix a gap of size $g = g_n = \lfloor e^{h_0 n/4} \rfloor$.

Lemma 10. *For μ-almost every $\underline{y}$ we have $\mathbb{E}(G_n) \to 0$.*

Proof. By stationarity of μ we obtain

$$\begin{aligned}\mathbb{E}(G_n) &= \sum_{i=1}^{k_n}\int_{\Omega}\mu_{\theta^i\omega}(A\cap\{R(\cdot,A)\le g\})\,d\mathbb{P}(\omega)\\ &= k_n\,\mu(A\cap\{R(\cdot,A)\le g\}) = k_n\,\mu(A)\,\mu(R(\cdot,A)\le g|A),\end{aligned} \tag{3}$$

where $\mu(\cdot|A)$ stands for the usual conditional measure on A. Using relation (3) above and that $k_n\mu(A) \to t$ we are left to prove that $\mu(R(\cdot, A) \le g|A) \to 0$ (as n tends to infinity) for μ-almost every $\underline{y}$.

By the Ornstein-Weiss theorem [22] we have for ν-a.e. $(\omega, \underline{x})$

$$\lim_{n\to\infty} \frac{1}{n} \log R_n(\underline{x}, \underline{x}) = h > h_0/2 > 0,$$

since by our assumptions $h_0 \le h$. Given $n_0 \ge 1$ consider the set $D(n_0) = \{\underline{x} \in X \colon R_n(\underline{x}, \underline{x}) \ge e^{nh_0/2} \text{ for all } n \ge n_0\}$.

Let $\varepsilon > 0$ be small and fixed. Since $\mu(D(n_0)) = \nu(\Omega \times D(n_0))$ goes to 1 as $n_0 \to \infty$, we can take n_0 so large that $\mu(D(n_0)) > 1 - \varepsilon$. Let $\underline{y}$ be a Lebesgue density point of $D(n_0)$ for the measure μ. It holds that

$$\mu(D(n_0)|C^n(\underline{y})) := \frac{1}{\mu(C^n(\underline{y}))}\, \mu(C^n(\underline{y}) \cap D(n_0)) \ge 1 - \varepsilon$$

for all large n. Therefore

$$\mu(R(\cdot, C^n(\underline{y})) \le g|C^n(\underline{y})) \le \mu(D(n_0)^c|C^n(\underline{y})) < \varepsilon.$$

Hence, taking a sequence $\varepsilon_q \to 0$ gives the conclusion. □

Lemma 11. *For μ-a.e $\underline{y} \in X$ we have $G_n \to 0$ $\mathbb{P}$-almost surely.*

Proof. Reproducing the computation of the second moment in the proof of Lemma 9, with the same m_n, taking into account that now $A \cap \{R(\cdot, A) \le g\} \in \mathcal{F}_0^{n+g}$ only, we get

$$\begin{aligned}\operatorname{var} G_n &= \sum_{i,j} \int_\Omega \mu_{\theta^i\omega}(A \cap \{R(\cdot, A) \le g\})\mu_{\theta^j\omega}(A \cap \{R(\cdot, A) \le g\})d\mathbb{P}(\omega) - \mathbb{E}(G_n)^2 \\ &\le 2c\mathbb{E}(G_n)me^{-h_0 n} + 2k_n^{1+\frac{3}{p}}\alpha(m-n-g) + C_0k_n^2\alpha(m-n-g) + \mathbb{E}(C^p)ce^{-h_0 n}.\end{aligned}$$

This proves that $\sum_n \operatorname{var} G_n < \infty$. The conclusion follows as in the proof of Lemma 9 using Lemma 10. □

Lemma 12. *For all $\underline{y} \in X'$ we have $H_n \to 0$ $\mathbb{P}$-almost surely.*

Proof. We use the correlation hypothesis (III) to obtain

$$H_n(\omega) \le \sum_{i=1}^{k_n} C(\theta^i\omega)\alpha(g-n) \le \alpha(g-n)\left(\sum_{i=1}^{k_n} C(\theta^i\omega)^p\right)^{1/p}.$$

By the ergodic theorem we have in addition $\sum_{i=1}^{k_n} C(\theta^i\omega)^p = O(k_n\mathbb{E}(C^p))$ for $\mathbb{P}$-almost every ω. Consequently

$$H_n \le \alpha(g-n)O\left(k_n^{\frac{1}{p}}\right) \to 0 \quad \mathbb{P}\text{-almost surely.}$$

□

Lemma 13. *For all $\underline{y} \in X'$ we have $K_n \to 0$, $\mathbb{P}$-almost surely.*

Proof. We have $K_n \le gce^{-h_0 n}M_n$ since

$$\mu_\omega(R(\cdot, A) \le g) = \mu_\omega\Big(\bigcup_{i=1}^{g} \sigma^{-i}A\Big) \le \sum_{i=1}^{g} \mu_{\theta^i\omega}(A) \le gce^{-h_0 n}.$$

In addition, M_n converges $\mathbb{P}$-almost surely by Lemma 9, which gives the conclusion. $\square$

We are now in a position to finish the proof of our first main result.

Proof of Theorem 1. By Lemmas 11,12 and 13 we have $G_n+H_n+K_n \to 0$ $\mathbb{P}$-almost surely and for μ-almost every $\underline{y}$. The theorem follows then from Lemma 6 using Lemma 8. $\square$

5. Random endomorphisms with decay of correlations

This section is devoted to the proof of Theorem 4 on random dynamical systems. Since the strategy is analogous to the one of Section 4 we will only write the proofs of the versions of Lemmas 9 and 10 with full detail and leave the adaptations of the other lemmas to the reader.

Write $\Gamma(m) = \sum_{\ell=m}^{\infty} \gamma(\ell)$. Note that Γ is also super polynomially decreasing. Define the set X' by

$$X' = \{x \in X : \overline{d}_\mu(x) \le d_1\}$$

for some constant d_1 sufficiently large. Since the upper dimension is μ-a.e. bounded by the dimension of the space X itself, it suffices to take $d_1 = \dim(X)$ to get a full measure set.

We fix $t>0$, take $A = B(x,r)$ and set $M_r = M_{A,t}$ analogously to Section 3. We will also write k_r, G_r, H_r and K_r for simplicity.

Lemma 14. *For all $x \in X'$ we have $M_r \to t$ in probability on $(\Omega, \mathbb{P})$ as $r \to 0$.*

Proof. We use the same method as in Lemma 9. Recall that

$$M_r = \sum_{i=1}^{k_r} \mu_{\theta^i\omega}(B(x,r)).$$

We use a set Ω_r defined as Ω_n with k_r instead of k_n, and estimate $E(M_r^2 1_{\Omega_r})$. Take $m = m_r = \lfloor r^{-u} \rfloor$, u to be chosen later.

Those $|i-j| < m$ using (II') again give a contribution less than $2ctmr^{d_0}$.

Those $|i-j| \ge m$, using (I') and (III') give a contribution less than (below $\varphi_{x,r}$ denotes a ρ^{-1}-Lipschitz function on X such that $1_{B(x,r)} \le \varphi_{x,r} \le 1_{B(x,r+\rho)}$)

$$\begin{aligned}
&2\sum_{i=1}^{k_r-m}\sum_{j=i+m}^{k_r}\int_{\Omega_r}\mu_{\theta^i\omega}(B(x,r))\mu_{\theta^j\omega}(B(x,r))d\mathbb{P}(\omega)\\
&\le 2\sum_{i=1}^{k_r}\sum_{\ell=m}^{k_r}\int_{\Omega_r}\left[\int_X \varphi_{x,r}1_{B(x,r)}\circ f_\omega^\ell d\mu_{\theta^i\omega} + C(\theta^i\omega)\gamma(\ell)\rho^{-1}\right]d\mathbb{P}(\omega)\\
&\le 2\sum_{i=1}^{k_r}\sum_{g=m}^{k_r}\left[C_0\gamma(\ell)\rho^{-1} + \mu(B(x,r+\delta))\mu(B(x,r))\right] + \rho^{-1}\sum_{\ell=m}^{k_r}\gamma(\ell)\mathbb{E}(1_{\Omega_r}\sum_{k=1}^{k_r}C\circ\theta^i)\\
&\le C_0k_r\Gamma(m)\rho^{-1} + k_r^2\mu(B(x,r+\rho))\mu(B(x,r)) + \rho^{-1}\Gamma(m)k_r^{3/p}.
\end{aligned}$$

By assumption (IV') we have $\mu(B(x,r+\rho)) \le \mu(B(x,r)) + r^{-b}\rho^a$ therefore one can choose u such that the expectation $E(M_r^2 1_{\Omega_r}) \le r^c$ for some constant $c>0$.

On the other hand $E(M_r 1_{\Omega_r^c})$ satisfies the same upper bound that in Lemma 9, therefore the variance of M_r is bounded from above by r^c (changing the constant c is necessary).

This proves that the variance of M_r converges to zero as $r \to 0$, hence M_r itself converges to t in L^2, thus in probability. $\square$

Remark 15. *Indeed, since the variance of M_r is bounded by r^c a Borel-Cantelli argument as in the proof of Lemma 9 shows that $M_{r_k} \to t$ a.s. for the subsequence $r_k = a^k$, for any $a \in (0,1)$.*

We now set the gap to $g_r = \lfloor r^{-d_0/4} \rfloor$.

Lemma 16. *For μ-almost every x we have $\mathbb{E}(G_r) \to 0$.*

Proof. The proof follows the one of Lemma 10. We have by stationarity that

$$\begin{aligned}\mathbb{E}(G_r) &= k_r \int_\Omega \mu_\omega(A \cap \{\tau_A^\omega \leq g\}) d\mathbb{P}(\omega) \\ &= k_r \int_{\Omega\times X} 1_{B(x,r)}(y) 1_{\{\tau^\omega_{B(x,r)} \leq g\}}(y) d\nu(\omega, y).\end{aligned}$$

Using assumptions (I'), (II') and (V'), the random recurrence rate [21] gives

$$\liminf_{r\to 0} \frac{\log \tau_r^\omega(x)}{-\log r} \geq \underline{d}_\mu(x) \geq d_0$$

for ν-a.e. (ω, x). Let $r_0 > 0$. Set $\alpha = d_0/2$ and

$$Q(r_0, y) = \{\omega \in \Omega \colon \exists r < r_0, \tau^\omega_{2r}(y) < r^{-\alpha}\}.$$

Denote by $(\nu_y)_{y\in X}$ the decomposition of the measure ν on X, meaning

$$\nu(E) = \int_X \nu_y(\{\omega \in \Omega \colon (\omega, y) \in E\}) d\mu(y),$$

for all measurable $E \subset \Omega \times X$. Let $\eta > 0$ and set

$$E_\eta(r_0) = \{y \in X \colon \nu_y(Q(r_0, y)) \leq \eta\}.$$

Let x be a Lebesgue density point of the set $E_\eta(r_0)$ for the measure μ, i.e.

$$\frac{\mu(B(x,r) \cap E_\eta(r_0))}{\mu(B(x,r))} \to 1$$

as $r \to 0$. Hence there exists $r_1 < r_0$ such that for any $r < r_1$

$$\mu(B(x,r) \cap E_\eta(r_0)^c) \leq \eta\mu(B(x,r)).$$

Let $r < r_1$. Since $g < r^{-\alpha}$ we get

$$\begin{aligned}\mathbb{E}(G_r)/k_r &= \int_{\Omega\times X} 1_{B(x,r)}(y) 1_{\{\tau^\omega_{B(x,r)}(y) \leq g\}} d\nu(\omega, y) \\ &\leq \int_{\Omega\times X} 1_{B(x,r)}(y) 1_{\{\tau^\omega_{2r}(y) < r^{-\alpha}\}} d\nu(\omega, y) \\ &\leq \int_{\Omega\times X} 1_{B(x,r)}(y) 1_{\{Q(r_0,y)\}}(\omega) d\nu(\omega, y) \\ &= \int_X 1_{B(x,r)}(y) \nu_y(Q(r_0, y)) d\mu(y) \\ &\leq \mu(B(x,r) \cap E_\eta(r_0)^c) + \eta\mu(B(x,r) \cap E_\eta(r_0)) \\ &\leq 2\eta\mu(B(x,r)).\end{aligned}$$

Since η is arbitrary and the measure of $E_\eta(r_0)$ can be made arbitrarily close to one, this shows that $E(G_r) \to 0$ for μa.e. x, since $k_r\mu(B(x,r)) \to t$. $\square$

Finally, using Lemma 16 and Markov's inequality, we obtain that G_r converges to zero in probability. The proofs that H_r and K_r converge to zero a.s. ω may be proven exactly as in the previous section so we do not add the details. This proves Theorem 4.

Remark 17. *If we strengthen assumptions (I') and (III') to allow functions φ which are dynamically Lipschitz, such that $B(x,r) \cap \{\tau_{B(x,r)} \leq g_r\}$ may be well approximated by these functions, then the proof of Lemma 11 may be adapted to this setting and give the a.s. convergence for all sequences $r_n = a^n$ (see also Remark 15). This strategy should work for example for the class of random unimodal maps as studied in* [3].

6. Examples

In this section we provide some examples that fulfill the hypotheses of our main theorems.

Example 18. *Let $s \geq 1$ and (Ω, θ) be a subshift of finite type on the symbolic space $\{0,1,\dots,s\}^{\mathbb{Z}}$ endowed with the distance $d_\Omega(\omega,\tilde{\omega}) = \sum_{n\in\mathbb{Z}} 2^{-|n|}|\omega_i - \tilde{\omega}_i|$. Let $\mathbb{P}$ be a Gibbs measure from a Hölder potential.*

Let $b \geq 1$ and make the shift $\{0,1,\dots,b\}^{\mathbb{N}}$ a random subshift by putting on it the random Bernoulli measures constructed as follows. Let $W = (w_{ij})$ be a $s\times b$ stochastic matrix with entries in $(0,1)$ and set $q = \max(w_{ij})$. Set $p_j(\omega) = w_{\omega_0,j}$. The random Bernoulli measure μ_ω is defined by $\mu_\omega([x_0 \dots x_n]) = p_{x_0}(\omega)p_{x_1}(\theta\omega)\dots p_{x_n}(\theta^n\omega)$. Since μ_ω are Bernoulli measures, one can observe easily that for all m,n, $A \in \mathcal{F}_0^n$ and $B \in \mathcal{F}_0^m$:

$$\left|\mu_\omega(A \cap \sigma^{-g-n}B) - \mu_\omega(A)\mu_{\theta^{n+g}\omega}(B)\right| = 0 \tag{4}$$

for every $g \geq 1$ and every $\omega \in \Omega$. Thus, property (III) is satisfied.

Moreover, we obtain that for every cylinder $[x_0 \dots x_n]$ and $\omega \in \Omega$

$$\mu_\omega([x_0 \dots x_n]) \leq q^{n+1}$$

for all $n \geq 1$, which implies property (II).

Now we will prove that property (I) holds for the marginal probability measure $\mu = \int_\Omega \mu_\omega \, d\mathbb{P}$. The proof explores the mixing properties of the base dynamics $\theta : \Omega \to \Omega$. In order to estimate the decay for the integrated measure μ we write, for m,n, $A \in \mathcal{F}_0^n$ and $B \in \mathcal{F}_0^m$:

$$\begin{aligned}\left|\mu(A\cap\sigma^{-n-g}B) - \mu(A)\mu(B)\right| &= \left|\int_\Omega \mu_\omega(A\cap\sigma^{-n-g}B)d\mathbb{P} - \int_\Omega \mu_\omega(A)d\mathbb{P}\int_\Omega \mu_\omega(B)d\mathbb{P}\right| \\ &\leq \int_\Omega \left|\mu_\omega(A\cap\sigma^{-n-g}B) - \mu_\omega(A)\mu_{\theta^{n+g}\omega}(B)\right| d\mathbb{P} \\ &\quad + \left|\int_\Omega \mu_\omega(A)\mu_{\theta^{n+g}\omega}(B)d\mathbb{P} - \int_\Omega \mu_\omega(A)d\mathbb{P}\int_\Omega \mu_\omega(B)d\mathbb{P}\right|\end{aligned}$$

for all $g \geq 1$. Using (4), the first term in the right hand side above is null. To control the second term consider the partition of Ω given by sets $U \cap \theta^{-n-g}V$, where U, V are cylinders of rank n, m respectively. By definition of the Bernoulli measure, the value $\mu_\omega(A)$ for $\omega \in U$ is constant equal to say $\eta_{U,A}$. We denote analogously by

$\eta_{V,B}$ the value taken by $\mu_\omega(B)$ for $\omega \in V$. The Gibbs measure $\mathbb{P}$ is exponentially ψ-mixing in the sense that there exists a function ψ such that

$$\left|\mathbb{P}(U \cap \theta^{-n-g}V) - \mathbb{P}(U)\mathbb{P}(V)\right| \le \psi(g)\mathbb{P}(U)\mathbb{P}(V),$$

with $\psi(g) \to 0$ exponentially fast. Writing

$$\int_\Omega \mu_\omega(A)\mu_{\theta^{n+g}\omega}(B)d\mathbb{P}(\omega) = \sum_{U,V} \eta_{U,A}\eta_{V,B}\mathbb{P}(U \cap \theta^{-n-g}V)$$

and

$$\int_\Omega \mu_\omega(A)d\mathbb{P}(\omega) = \sum_U \eta_{U,A}\mathbb{P}(U), \quad \int_\Omega \mu_\omega(B)d\mathbb{P}(\omega) = \sum_V \eta_{V,B}\mathbb{P}(B)$$

we get that

$$\left|\mu(A \cap \sigma^{-n-g}B) - \mu(A)\mu(B)\right| \le \sum_{U,V} \mathbb{P}(U)\mathbb{P}(V)\psi(g) \le \psi(g),$$

which decays exponentially fast with g (independently of m and n) and property (I) holds.

Therefore, it follows from our results that for μ-almost every $\underline{y}$, $\mathbb{P}$-almost every ω and all $t \ge 0$ we have

$$\mu_\omega\left(\underline{x} \in X \colon R_n(\underline{x}, \underline{y}) > \frac{t}{\mu(C^n(\underline{y}))}\right) \to e^{-t}, \quad as\ n \to \infty.$$

and that for μ-almost every $y \in X$ and all $t \ge 0$ we have

$$\mu\left(x \in X \colon R_n(\underline{x}, \underline{y}) > \frac{t}{\mu(C^n(\underline{y}))}\right) \to e^{-t}, \quad as\ n \to \infty.$$

The next example shows that the sample measures μ_ω and the marginal μ can be mutually singular for a.e. ω, as announced in Section 2. This is a special case of Example 18.

Example 19. *Let $\Omega = \{0,1\}^{\mathbb{Z}}$ with the shift θ and the Bernoulli measure $\mathbb{P}$ with weights $(1/2, 1/2)$. Make the shift $\{0,1\}^{\mathbb{N}}$ a random shift by putting on it the random Bernoulli measures constructed as follows. Take $p \in (0, 1/2)$ and $q = 1-p$. Set $p_0(\omega) = p$ if $\omega_0 = 0$ and q otherwise, and $p_1(\omega) = 1 - p_0(\omega)$. The random Bernoulli measure μ_ω is defined by $\mu_\omega([x_0 \dots x_n]) = p_{x_0}(\omega)p_{x_1}(\theta\omega)\dots p_{x_n}(\theta^n\omega)$. Indeed $\mu_\omega([x_0 \dots x_n]) = p^{k_n(\omega,x)}q^{n-k_n(\omega,x)}$ where $k_n(\omega, x)$ is the number of $i = 0, \dots, n$ such that $\omega_i = x_i$. The marginal measure μ is*

$$\mu([x_0 \dots x_n]) = \int_\Omega \mu_\omega([x_0 \dots x_n])d\mathbb{P}(\omega) = \int p_{x_0}(\omega)d\mathbb{P}(\omega)\dots\int_\Omega p_{x_n}(\theta^n\omega)d\mathbb{P}(\omega)$$

since each $p_{x_k}(\theta^k\omega)$ depends only on ω_k and the ω_k's are independent. Moreover each of these integrals is equal to $(p+q)/2 = 1/2$. Therefore μ is the Bernoulli measure on Ω with weights $(1/2, 1/2)$. Next, if the density ρ_ω of the probability measure μ_ω with respect to μ exists on a set of positive measure then the limit of the ratio

$$\mu_\omega([x_0 \dots x_n])/\mu([x_0 \dots x_n]) = p^{k_n(\omega,x)}q^{n-k_n(\omega,x)}2^n$$

should exists μ-almost everywhere on this set, by Lebesgue differentiation theorem, and should be equal to $\rho_\omega(x)$. However, since k_n has increments 0 or 1 the only possible limits of the ratio are 0 or ∞ since p and q are not equal to $1/2$. Since the density ρ_ω cannot have finite nonzero value, therefore it does not exist.

Note that our results extend to the random dynamical systems context some results obtained for in the deterministic setting in [27], in which case we take $\mathbb{P}$ to be a Dirac measure at a fixed point for θ. In particular we obtain applications to the thermodynamical formalism of random dynamical systems. Let $S : \mathcal{E} \to \mathcal{E}$ be as before the skew-product given by $S(\omega, x) = (\theta(\omega), \sigma(x))$. Given a measurable potential $\phi : \mathcal{E} \to \mathbb{R}$ set $\operatorname{var}_n \phi(\omega) = \sup\{|\phi(\omega, x) - \phi(\omega, \tilde{x})| : x_i = \tilde{x}_i \text{ for all } i < n\}$. If ϕ satisfies $\int \sup_x |\phi(\omega, x)|\, d\mathbb{P}(\omega) < \infty$ and $\operatorname{var}_n \phi(\omega) \le K_\phi(\omega) e^{-\tau n}$ for all $n \ge 1$, for some random variable with $\log K_\phi \in L^1(\mathbb{P})$, then the variational principle holds

$$\pi_S(\phi) = \sup_\eta \Big\{ h_\eta(S) + \int \phi \, d\eta \Big\},$$

where the supremum is taken over all S-invariant probability measures and $\pi_S(\phi)$ denotes the topological pressure of S with respect to ϕ c.f. [5, 18]. We say that an S-invariant probability measure μ is an *equilibrium state for S with respect to ϕ* if it attains the previous supremum. In addition, we say that a probability measure μ that admits a disintegration $(\mu_\omega)_\omega$ is a *fiber Gibbs measure with respect to ϕ* if there exist random variables $\lambda = \lambda(\omega)$, $C_\phi = C_\phi(\omega) > 0$ such that $\int \log C_\phi(\omega)\, d\mathbb{P}(\omega) < \infty$ and

$$C_\phi(\omega)^{-1} \le \frac{\mu_\omega(C_n(\underline{x}))}{\exp\big(-\log \prod_{j=0}^{n-1} \lambda(\theta^j(\omega)) + \sum_{j=0}^{n-1} \phi(S^j(\omega, \underline{x}))\big)} \le C_\phi(\omega) \tag{5}$$

for $\mathbb{P}$-a.e. ω, every $\underline{x} \in \mathcal{E}_\omega$ and $n \ge 1$. In fact, under the previous conditions, it follows from [18, Theorem 2.1] that there exists a unique equilibrium state for S with respect to ϕ and that it is a fiber Gibbs measure. Although in general the measure of cylinders may decay exponentially to zero but not uniformly in ω we build an example below where this is not the case. Let us mention that in most of the known results the thermodynamical formalism follows from a carefull analysis of transfer operators. Given $\omega \in \Omega$ the associated random Perron-Frobenius operator is

$$(\mathcal{L}^\omega g)(x) = \sum_{S(\omega, y) = x} e^{\phi(\omega, y)} g(y) \tag{6}$$

and, in our context, for every continuous $g : \mathcal{E}_\omega \to \mathbb{R}$ it defines a continuous function $\mathcal{L}^\omega g$ on $\mathcal{E}_{\theta(\omega)}$. Set $\mathcal{L}^{\omega,n} = \mathcal{L}^{\theta^{n-1}(\omega)} \circ \cdots \circ \mathcal{L}^{\theta(\omega)} \circ \mathcal{L}^\omega$ for all $n \ge 0$.

Let us mention that these results also hold for the random composition of any finite number of uniformly expanding maps as in Theorem 4.

Example 20. *Set $\Omega = \{0,1\}^{\mathbb{Z}}$ with the distance $d_\Omega(\omega, \tilde{\omega}) = \sum_{n \in \mathbb{Z}} 2^{-n} |\omega_i - \tilde{\omega}_i|$. Let $f_0, f_1 : \mathbb{T}^d \to \mathbb{T}^d$ be C^2-smooth expanding maps and d denote the product metric in $\Omega \times \mathbb{T}^d$. Assume that $\mathbb{P}$ is a Bernoulli measure on Ω as before and consider the potential $\phi : \mathcal{E} \to \mathbb{R}$ given by $\phi(\omega, x) = -\log|\det Df_{\omega_0}(x)|$, which is piecewise constant. By the change of variables formula we obtain that the probability measures $\nu_\omega = Leb$ are conformal, in the sense that $(\mathcal{L}^\omega)^* \nu_\omega = \nu_{\theta(\omega)}$. Moreover, it follows from* [18, Theorem 2.1] *that there exists a measurable family of continuous and integrable functions $(h_\omega)_\omega$ such that $\mathcal{L}^\omega h_\omega = \lambda_\omega h_{\theta(\omega)}$, that $\int h_\omega \, d\nu_\omega = 1$ and that $(\mu_\omega)_\omega$ given by $d\mu_\omega = h_\omega d\nu_\omega$ is the unique S-invariant probability measure that is an equilibrium state for S with respect to ϕ.*

Now, since f_ω is either f_0 or f_1 (finite number of functions) then the potential ϕ is locally constant and the family $(\mathcal{L}_\omega)_\omega$ of transfer operators reduce to finitely *many of them. Consequently, there are uniform constants $L > 1$ and $\varepsilon > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$ the cones*

$$\Lambda_L^\omega = \Big\{ g > 0 : g(x) \le g(y) \exp(L d(x,y)^\alpha)) \text{ for all } d(x,y) < \varepsilon \Big\}$$

of continuous functions are strictly preserved by the random Perron-Frobenius operator for all positive iterates. More precisely, for all $n \ge 1$ one has $\mathcal{L}^{\omega,n}(\Lambda_L^\omega) \subset \Lambda_{2L/3}^{\theta^n(\omega)}$ (c.f. [18, Equations (4.9) and (4.11)-(4.19)]*). Since each function h_ω belongs to the cone Λ_L^ω of observables and also $\int h_\omega d\nu_\omega = 1$ it holds that these have uniform Hölder constants.*

In fact, we use the fact that $h_\omega = \lim_{n\to\infty} \mathcal{L}^{\theta^{-n}\omega,n} 1$ and the speed of convergence is exponential, meaning that there exists $a > 0$ so that

$$\|h_\omega - \lim_{n\to\infty} \mathcal{L}^{\theta^{-n}\omega,n} 1\| \le e^{-an} \tag{7}$$

for all $n \ge 1$ large, to prove that $\Phi_\omega = \int \varphi h_\omega dLeb$ varies Hölder continuously with ω. For completeness let us mention that equation (7) *above corresponds to* [18, Equation 4.43] *with $C_\omega = C_0$ constant. Now, if one assumes ω and ω' are in the same $-2n, \dots, 2n$ cylinder then*

$$\begin{aligned} |\Phi_\omega - \Phi_{\omega'}| &\le \left(\int \varphi dLeb \right) \sup_X |h_\omega - h_{\omega'}| \\ &\le \left(\int \varphi dLeb \right) \left(\sup_X |\mathcal{L}^{\theta^{-n}\omega,n} 1 - \mathcal{L}^{\theta^{-n}\omega',n} 1| + 2e^{-an} \right). \end{aligned}$$

Since $\theta^{-n}\omega$ and $\theta^{-n}\omega'$ are in the same $[-n..2n]$ cylinder and $\mathcal{L}^\omega$ is locally constant then then the first summand in the right hand side above is null, leading to $|\Phi_\omega - \Phi_{\omega'}| \le 2e^{-an}$ thus proving the claim that $\omega \mapsto \Phi_\omega$ is Hölder continuous.

Moreover, since each function h_ω belongs to the cone Λ_L^ω of observables and also $\int h_\omega d\nu_\omega = 1$ there exists a uniform constant $\tilde{K} \ge 1$ (depending only on ϵ) such that $|h_\omega(x)| \le \tilde{K} e^{L\, diam(M)^\alpha}$ and consequently there exists a uniform constant $K > 0$ such that $\|h_\omega\|_\infty \le K$ for $\mathbb{P}$-a.e. ω. In consequence,

$$\mu_\omega(B(x,r)) \le K Leb(B(x,r)) \le c r^d$$

and, since $\mu = \int \mu_\omega d\mathbb{P}$, then μ is also absolutely continuous with respect to Leb with densitiy bounded by K. As a consequence we get for all small r, ρ that

$$\mu(B(x, r+\rho)) \le c(r+\rho)^d \le \mu(B(x,r)) + r^{-1}\rho.$$

This proves that (II') and (IV') hold.

We are now left to discuss the mixing properties (I') and (III'). The fiber mixing property (III') is a consequence of Theorem 2.2 and Equation 5.19 in [18] *that there exists a random variable C so that*

$$\left| \int_X \varphi\,(\psi \circ f_\omega^\ell)\, d\mu_\omega - \int_X \varphi\, d\mu_\omega \int_X \psi\, d\mu_{\theta^\ell \omega} \right| \le C(\omega)\gamma(\ell) Lip(\varphi) \sup |\psi|.$$

We refer the reader to Lemma 6.3 in [19] *for the precise estimates leading to the previous expression. Furthermore, since there are finitely many expanding maps, one can check in Section 4 and Section 5 of* [18] *that the random variable C can be taken bounded from above by a uniform constant C_0.*

Now we will prove that property (I') holds for the marginal probability measure $\mu = \int \mu_\omega\, d\mathbb{P} = \int h_\omega dLeb\, d\mathbb{P}$. The proof explores the mixing properties of $\theta : \Omega \to \Omega$.

In order to estimate the decay for the integrated measure $\mu = \int \mu_\omega \, d\mathbb{P}$ *we write for all* $n \geq 1$

$$\begin{aligned}
&\Big| \int \varphi(\psi \circ f_\omega^\ell)\, d\mu - \int \varphi \, d\mu \int \psi \, d\mu \Big| \\
&\quad = \Big| \int \int \varphi(\psi \circ f_\omega^\ell)\, d\mu_\omega d\mathbb{P} - \int \varphi \, d\mu_\omega d\mathbb{P} \int \psi \, d\mu_{\theta^\ell(\omega)} d\mathbb{P} \Big| \\
&\quad \leq \int \Big| \int \varphi(\psi \circ f_\omega^\ell)\, d\mu_\omega - \int \varphi \, d\mu_\omega \int \psi \, d\mu_{\theta^\ell(\omega)} \Big| \, d\mathbb{P} \qquad (8)\\
&\quad + \Big| \int \Big(\int \varphi \, d\mu_\omega \int \psi \, d\mu_{\theta^\ell(\omega)} \Big) d\mathbb{P} - \int \varphi \, d\mu \int \psi \, d\mu \Big| . \qquad (9)
\end{aligned}$$

On the one hand, that by (III') the term in equation (8) *is bounded from above by* $C_0 \, \gamma(\ell) \sup(\psi) \|\varphi\|_\alpha$. *On the other hand, by the exponential decay of correlations for the shift* θ *we get, if one considers the observables* $\Phi_\omega = \int \varphi d\mu_\omega = \int \varphi h_\omega d Leb$ *and* $\Psi_\omega = \int \psi d\mu_\omega = \int \psi h_\omega d Leb$ *then the expression in equation* (9) *is such that*

$$\begin{aligned}
\Big| \int \Phi_\omega \, \Psi_{\theta^\ell(\omega)} d\mathbb{P} - \int \Phi_\omega \, d\mathbb{P} \int \Psi_\omega d\mathbb{P} \Big| &\leq K \sup(\Psi_\omega) \|\Phi_\omega\|_\alpha \gamma(\ell) \\
&\leq K \sup(\Psi) \|\Phi\|_\alpha \gamma(\ell)
\end{aligned}$$

for some positive constant K *and all* $\ell \geq 1$. *This proves that condition (I') holds.*

Finally, let us prove the random aperiodicity condition (V'). We observe that

$$\begin{aligned}
\nu((\omega, x) \in \mathcal{E} \colon \exists n \in \mathbb{N} \;\; f_\omega^n(x) = x) &\leq \sum_{n \geq 1} \nu((\omega, x) \in \mathcal{E} \colon f_\omega^n(x) = x) \\
&\leq \sum_{n \geq 1} \int_\Omega \mu_\omega \Big(x \in X \colon f_\omega^n(x) = x \Big) \, d\mathbb{P}
\end{aligned}$$

and, since $\mu_\omega \ll Leb$, *it is enough to prove that* $Leb(x \in X \colon f_\omega^n(x) = x) = 0$ *for* $\mathbb{P}$*-almost every* ω. *This property follows immediately from the fact that* f_ω^n *is an expanding map, since the periodic points of all periods are isolated and thus finite. This proves that* $S : \mathcal{E} \to \mathcal{E}$ *is random aperiodic as claimed.*

Therefore, it follows from our Theorem 4 that for μ*-almost every* $\underline{y}$ *and all* $t \geq 0$ *we have*

$$\mu_\omega \left(x \in X \colon \tau^\omega_{B(y,r)}(x) > \frac{t}{\mu(B(y,r))} \right) \xrightarrow{\mathbb{P}} e^{-t}, \quad \textit{as } n \to \infty.$$

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