RECURRENCE RATES AND HITTING-TIME DISTRIBUTIONS FOR RANDOM WALKS ON THE LINE

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ABSTRACT. We consider random walks on the line given by a sequence of independent identically distributed jumps belonging to the strict domain of attraction of a stable distribution, and first determine the almost sure exponential divergence rate, as $\varepsilon \to 0$, of the return time to $(-\varepsilon, \varepsilon)$. We then refine this result by establishing a limit theorem for the hitting-time distributions of $(x-\varepsilon, x+\varepsilon)$ with arbitrary $x \in \mathbb{R}$.

1. INTRODUCTION AND RESULTS

We consider a recurrent random walk on $\mathbb{R}$, $S_0 := 0$ and $S_n := X_1 + \cdots + X_n$, $n \geq 1$, where the X_i are i.i.d. random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\frac{S_n}{A_n}$ converges, for positive real numbers A_n, in distribution to a stable random variable X with index α. Necessarily (due to recurrence), $\alpha \in [1, 2]$, and the sequence $(A_n)_{n\geq 1}$ is regularly varying of index $\frac{1}{\alpha}$, satisfying $\sum_{n\geq 1} \frac{1}{A_n} = \infty$.

To capture the speed at which recurrence appears, it is possible to specify, for such a walk, some deterministic sequences (ε_n) such that $S_n \in (-\varepsilon_n, \varepsilon_n)$ infinitely often, or $S_n \notin (-\varepsilon_n, \varepsilon_n)$ eventually, almost surely. This classical question was addressed, for example, in [7] and [5], the results of which have recently been extended in [6].

Here, we are going to study the number of steps it takes to return to some small neighborhood of the origin (or to hit a different small interval for the first time). For related work on random walks in the plane, intimately related to the $\alpha = 1$ case of the present paper, we refer to [12].

As an additional standing assumption on our walk, we will always require the distribution of the jumps X_i to satisfy the Cramer condition

$$\limsup_{|t|\to\infty} |\mathbb{E}[e^{itX_1}]| < 1. \tag{1}$$

This readily implies, in particular, that the event $\Omega^* := \{S_n \neq 0 \ \forall n \geq 1\}$ has positive probability, and Ω^* has probability one if and only if no individual path returning to the origin has positive probability.

As a warm-up we first determine the a.s. rate at which the variables

$$\mathbf{T}_\varepsilon := \min\{n \geq 1 \ : \ |S_n| < \varepsilon\}, \quad \varepsilon > 0,$$

diverge on Ω^* as $\varepsilon \to 0$. Let $\beta \in [2, \infty]$ be the exponent conjugate to α, that is, $\alpha^{-1} + \beta^{-1} = 1$.

Theorem 1. *In the present setup,*

$$\lim_{\varepsilon\to 0} \frac{\log \mathbf{T}_\varepsilon}{\log \varepsilon} = -\beta \quad \text{a.s. on } \Omega^*. \tag{2}$$

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Our main objective then is to determine the precise order of magnitude, and to study the asymptotic distributional behaviour, as $\varepsilon \to 0$, of the more general hitting times of ε-neighbourhoods of arbitrary given points x on the line. We shall, in fact, do so for the walk $S'_n := S'_0 + S_n$, $n \geq 0$, with random initial position S'_0, independent of $(S_n)_{n\geq 0}$ and having an arbitrary fixed distribution P on $\mathbb{R}$. For any $x \in \mathbb{R}$ we thus let

$$\mathbf{T}^x_\varepsilon := \inf\{m \geq 1 \ : \ |S'_m - x| < \varepsilon\}$$

and $\Omega^*_x := \{S'_n \neq x \ \forall n \geq 1\}$. Outside Ω^*_x we clearly have $\lim_{\varepsilon\to 0} \mathbf{T}^x_\varepsilon = \min\{m \geq 1 \ : \ S'_m = x\}$.

It is convenient to state the results in terms of, and work with, the strictly increasing continuous function $G : [0, +\infty) \to [0, +\infty)$ with $G(0) = 0$ which affinely interpolates the values $G(n) = \sum_{k=1}^n \frac{1}{A_k}$, $n \geq 1$. We denote by G^{-1} its inverse function. Evidently, $G(n) = o(n)$. Moreover, by the direct half of Karamata's theorem (cf. Propositions 1.5.8 and 1.5.9a of [2]), G is regularly varying with index $\frac{1}{\beta}$, and satisfies

$$\frac{n}{A_n} = o(G(n)) \ \text{ if } \alpha = 1, \qquad \text{while} \qquad \frac{n}{A_n} \sim \frac{G(n)}{\beta} \ \text{ in case } \alpha \in (1, 2]. \tag{3}$$

We establish a result on convergence in distribution for $\varepsilon\, G(\mathbf{T}^x_\varepsilon)$ conditioned on Ω^*_x (while $\varepsilon\, G(\mathbf{T}^x_\varepsilon) \to 0$ outside this set). In the case $\alpha = 1$, the limit distribution is the same as for square integrable random walk on the plane, cf. [12]. Recall that X has a density f_X. For simplicity we set $\gamma := 2 f_X(0)\, \mathbb{P}(\Omega^*)$.

Theorem 2. *Assume that $\alpha = 1$, and fix any $x \in \mathbb{R}$. Conditioned on Ω^*_x, the variables $\varepsilon\, G(\mathbf{T}^x_\varepsilon)$ converge in law,*

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\gamma\varepsilon\, G(\mathbf{T}^x_\varepsilon) \leq t \mid \Omega^*_x\right) = \frac{t}{1+t} \qquad \forall t > 0.$$

For $\alpha \in (1, 2]$, different limits distributions arise, and we obtain convergence in law of $\mathbf{T}^x_\varepsilon$ to the $\frac{1}{\beta}$-stable subordinator at an independent exponential time:

Theorem 3. *Assume that $\alpha \in (1, 2]$, and fix any $x \in \mathbb{R}$. Conditioned on Ω^*_x, the variables $\varepsilon\, G(\mathbf{T}^x_\varepsilon)$ converge in law,*

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\Gamma\Big(\frac{1}{\beta}\Big)\frac{\gamma}{\beta}\, \varepsilon\, G(\mathbf{T}^x_\varepsilon) \leq t \,\middle|\, \Omega^*_x\right) = \Pr\left(\mathcal{E}\mathcal{G}^{1/\beta}_{1/\beta} \leq t\right) \qquad \forall t > 0,$$

or, equivalently,

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\left(\Gamma\Big(\frac{1}{\beta}\Big)\frac{\gamma}{\beta}\right)^\beta \frac{\mathbf{T}^x_\varepsilon}{G^{-1}(1/\varepsilon)} \leq t \,\middle|\, \Omega^*_x\right) = \Pr\left(\mathcal{E}^\beta \mathcal{G}_{1/\beta} \leq t\right) \qquad \forall t > 0,$$

where $\mathcal{E}$ and $\mathcal{G}_{1/\beta}$ are independent random variables, $\Pr(\mathcal{E} > t) = e^{-t}$, and $\mathcal{G}_{1/\beta}$ having the one-sided stable law of index $\frac{1}{\beta}$ with Laplace transform $\mathbb{E}[e^{-s\mathcal{G}_{1/\beta}}] = e^{-s^{1/\beta}}$, $s > 0$.

In particular, we have:

Corollary 1. *If $(X_n)_{n\geq 1}$ is an i.i.d. sequence of centered random variables with variance 1, satisfying the Cramer condition, and $x \in \mathbb{R}$, then*

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(2\mathbb{P}(\Omega^*_x)\, \varepsilon\sqrt{\mathbf{T}^x_\varepsilon} \leq t \,\middle|\, \Omega^*_x\right) = \Pr\left(\frac{\mathcal{E}}{|\mathcal{N}|} \leq t\right) \qquad \forall t > 0,$$

or, equivalently,

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(4\mathbb{P}(\Omega^*_x)^2\, \varepsilon^2 \mathbf{T}^x_\varepsilon \leq t \mid \Omega^*_x\right) = \Pr\left(\left(\frac{\mathcal{E}}{|\mathcal{N}|}\right)^2 \leq t\right) \qquad \forall t > 0,$$

where $\mathcal{E}$ and $\mathcal{N}$ are independent variables, $\mathcal{N}$ having a standard Gaussian distribution $\mathcal{N}(0,1)$.

As Cheliotis does in [6], we will use the following extension of Stone's local limit theorem [13].

Proposition 1. *Let θ be such that $\limsup_{|t|\to\infty} |\mathbb{E}[e^{itX_1}]| < \theta < 1$, and let $c > 1$. Then there exist a real number $h_0 > 0$ and an integer $n_0 \geq 1$ such that, for any $n \geq n_0$, for any interval I contained in $[-h_0, h_0]$, of length larger than θ^n, we have*

$$c^{-1} f_X(0)|I| < \mathbb{P}\left(\frac{S_n}{A_n} \in I\right) < c f_X(0)|I|.$$

2. Almost sure convergence : proof of theorem 1

Proof of Theorem 1. To begin with, choose θ, c, and h_0 as in Proposition 1.

To first establish an estimate from below, we fix any $\xi > 1$ and set $\varepsilon_n := G(n)^{-\xi}$. This makes the series $\sum_n \mathbb{P}(|S_n| < \varepsilon_n)$ summable: Indeed, by regular variation and (3), we have $\frac{\varepsilon_n}{A_n} > \theta^n$ for n large, while

$$\frac{\varepsilon_n}{A_n} = O\left(\frac{G(n) - G(n-1)}{G(n-1)^\xi}\right) = O\left(\int_{n-1}^{n} \frac{G'(t)}{G(t)^\xi}\, dt\right),$$

which is summable since $\int_1^\infty \frac{G'(t)}{G(t)^\xi}\, dt = \left[\frac{G(t)^{1-\xi}}{1-\xi}\right]_1^\infty < \infty$. In particular, $(\frac{-\varepsilon_n}{A_n}, \frac{\varepsilon_n}{A_n}) \subseteq [-h_0, h_0]$ for large n. Proposition 1 therefore applies to these intervals, and shows that $\mathbb{P}(|S_n| < \varepsilon_n) = O(\frac{\varepsilon_n}{A_n})$ is summable as well. Hence, by the Borel-Cantelli lemma, $\mathbb{P}(|S_n| < \varepsilon_n \text{ i.o.}) = 0$. Since $\varepsilon_n \searrow 0$, we can conclude that $\mathbf{T}_{\varepsilon_n} > n$ eventually, almost surely on Ω^*, and we get $\liminf_{n\to\infty} \frac{\log G(\mathbf{T}_{\varepsilon_n})}{-\log \varepsilon_n} \geq \frac{1}{\xi}$ a.s. on Ω^*. Using monotonicity of $\log G(\mathbf{T}_\varepsilon)$ and the fact that $\varepsilon_{n+1} \sim \varepsilon_n$, this extends from the ε_n to the full limit as $\varepsilon \to 0$, and since $\xi > 1$ was arbitrary, we conclude that

$$\liminf_{\epsilon\to 0} \frac{\log G(\mathbf{T}_\varepsilon)}{-\log \varepsilon} \geq 1 \quad \text{a.s. on } \Omega^*. \tag{4}$$

To control the corresponding $\limsup$, we now fix any $\xi \in (0,1)$. From Proposition 1, using intervals $(\frac{-\varepsilon_n}{A_n}, \frac{\varepsilon_n}{A_n})$ and regular variation of $(A_n)_{n\geq 1}$, we see that there exists a constant $c' > 0$ such that for every $\varepsilon \in (0,1)$ there is some m_ε satisfying

$$\mathbb{P}(|S_k| < \varepsilon) \geq \frac{c'\varepsilon}{A_k} \qquad \text{for } k \geq m_\varepsilon.$$

More precisely, the dependence of m_ε on ε comes from the requirement $2\varepsilon/A_k > \theta^k$ for $k \geq m_\varepsilon$ on the length of intervals, which is met by taking $m_\varepsilon := \kappa(-\log\varepsilon)$ with a suitable constant $\kappa > 0$. Next, choose integers n_ε in such a way that $G(n_\varepsilon) \leq \varepsilon^{-\frac{1}{\xi}} < G(n_\varepsilon + 1)$. Inspired by a decomposition used by Dvoretski and Erdös [8], we consider the pairwise disjoint events $E_k^\varepsilon := \{|S_k| < \varepsilon \text{ and } \forall j = k+1, \ldots, n_\varepsilon : |S_j - S_k| > 2\varepsilon\}$, $1 \leq k \leq n_\varepsilon$. By independence and stationarity we have

$$1 \geq \sum_{k=m_\varepsilon}^{n_\varepsilon} \mathbb{P}(E_k^\varepsilon) \geq \sum_{k=m_\varepsilon}^{n_\varepsilon} \mathbb{P}(|S_k| < \varepsilon)\, \mathbb{P}(\mathbf{T}_{2\varepsilon} > n_\varepsilon - k) \geq c'\varepsilon\, \mathbb{P}(\mathbf{T}_{2\varepsilon} > n_\varepsilon) \sum_{k=m_\varepsilon}^{n_\varepsilon} \frac{1}{A_k}.$$

Combining this with $G(m_\varepsilon) = o(G(n_\varepsilon))$ (note that $G(m_\varepsilon)$ is slowly varying), we obtain

$$\mathbb{P}(G(\mathbf{T}_{2\varepsilon}) > \varepsilon^{-\frac{1}{\xi}}) \leq \mathbb{P}(G(\mathbf{T}_{2\varepsilon}) > G(n_\varepsilon)) = \mathbb{P}(\mathbf{T}_{2\varepsilon} > n_\varepsilon) \leq \frac{1}{c'\varepsilon\,(G(n_\varepsilon) - G(m_\varepsilon))} \sim \frac{\varepsilon^{\frac{1}{\xi}-1}}{c'}.$$

Therefore, if we let $\varepsilon_p := p^{-\frac{2}{1-\xi}}$, $p \geq 1$ the Borel-Cantelli lemma implies $G(\mathbf{T}_{2\varepsilon_p}) \leq \varepsilon_p^{-\frac{1}{\xi}}$ eventually almost surely, showing that $\limsup_{p\to+\infty} \frac{\log G(\mathbf{T}_{2\varepsilon_p})}{-\log(2\varepsilon_p)} \leq \frac{1}{\xi}$. Using monotonicity as before, we can extend this from the ε_p to the full limit $\varepsilon \to 0$, and since this is true for any $\xi \in (0,1)$, we obtain

$$\limsup_{\varepsilon\to 0} \frac{\log G(\mathbf{T}_\varepsilon)}{-\log(\varepsilon)} \leq 1 \quad \text{a.s. on } \Omega. \tag{5}$$

To conclude the proof, we note that for any $\alpha \in [1,2]$ we have

$$\lim_{n\to\infty} \frac{\log G(n)}{\log n} = \frac{1}{\beta},$$

which follows readily from regular variation of G (compare Fact 2 in [6]). Together with (4) and (5), this entails

$$\lim_{\varepsilon\to 0} \frac{\log \mathbf{T}_\varepsilon}{-\log \varepsilon} = \lim_{\varepsilon\to 0} \frac{\log \mathbf{T}_\varepsilon}{\log G(\mathbf{T}_\varepsilon)} \cdot \frac{\log G(\mathbf{T}_\varepsilon)}{-\log \varepsilon} = \beta \quad \text{a.s. on } \Omega^*,$$

as required. □

The first argument can easily be adapted to prove the lower bound (4) also for $\mathbf{T}_\varepsilon^x$ with $x \neq 0$.

3. Convergence in distribution for auxiliary processes

We need to introduce auxiliary processes. Let $(M_0^\varepsilon)_{\varepsilon>0}$ be a family of random variables, independent of $(S_n)_{n\geq 0}$, such that M_0^ε has uniform distribution on the interval $(-\varepsilon,\varepsilon)$. For each $\varepsilon > 0$ we define the walk $(M_n^\varepsilon)_{n\geq 0}$ with random initial position M_0^ε, that is, $M_n^\varepsilon := M_0^\varepsilon + S_n$.

A major step towards Theorems 2 and 3 will be to prove a version which applies to the variables

$$\tau_\varepsilon := \min\{n \geq 1 \colon |M_n^\varepsilon| < \varepsilon\}, \quad \varepsilon > 0.$$

That is, we are interested in the limiting behaviour, as $\varepsilon \to 0$, of the first return time distribution of the walk $(M_n^\varepsilon)_{n\geq 0}$ to the interval $(-\varepsilon,\varepsilon)$. The goal of the present section is to establish

Theorem 4. *Assume that $\alpha = 1$. Conditioned on Ω^*, the variables $\varepsilon\, G(\tau_\varepsilon)$ converge in law,*

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\gamma\varepsilon\, G(\tau_\varepsilon) \leq t \mid \Omega^*\right) = \frac{t}{1+t} \quad \forall t > 0. \tag{6}$$

Theorem 5. *Assume that $\alpha \in (1,2]$. Conditioned on Ω^*, the variables $\varepsilon\, G(\tau_\varepsilon)$ converge in law,*

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\Gamma\!\left(\frac{1}{\beta}\right)\frac{\gamma}{\beta}\,\varepsilon\, G(\tau_\varepsilon) \leq t \,\middle|\, \Omega^*\right) = \Pr\left(\mathcal{E}\mathcal{G}_{1/\beta}^{1/\beta} \leq t\right) \quad \forall t > 0. \tag{7}$$

Equivalently,

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\left(\Gamma\!\left(\frac{1}{\beta}\right)\frac{\gamma}{\beta}\right)^{\beta} \frac{\tau_\varepsilon}{G^{-1}(1/\varepsilon)} \leq t \,\middle|\, \Omega^*\right) = \Pr\left(\mathcal{E}^\beta\mathcal{G}_{1/\beta} \leq t\right) \quad \forall t > 0.$$

Again we start with considerations valid for any $\alpha \in [1,2]$. To begin with, we define, for $\varepsilon > 0$, $R > 0$, and integers $K > 0$, auxiliary events

$$\Gamma_{\varepsilon,R,K} := \{\forall i = 1,\ldots,K : S_i \neq 0 \text{ and } |M_i^\varepsilon| \leq R\},$$

which asymptotically exhaust Ω^*, and on which we can work conveniently. As $\varepsilon \to 0$ we have $\mathbb{P}(\Gamma_{\varepsilon,R,K}) \to \mathbb{P}(\Gamma_{R,K})$ and $\mathbb{P}(\Gamma_{\varepsilon,R,K} \setminus \Omega^*) \to \mathbb{P}(\Gamma_{R,K} \setminus \Omega^*)$, where $\Gamma_{R,K} := \{\forall i = 1,\ldots,K : 0 <$

$|S_i| \le R\}$ (except, perhaps, for a countable set of R's which we are going to avoid). Let $n \in \mathbb{N}$. Using again a decomposition similar to that of Dvoretski and Erdös in [8], we find, for $\varepsilon \in (0, \frac{1}{2})$,

$$\mathbb{P}(\Gamma_{\varepsilon,R,K}) = \sum_{k=0}^{n} p_k^- = \sum_{k=0}^{n} p_k^+ \tag{8}$$

with $p_k^\pm = p_{k,n,\varepsilon,R,K}^\pm := \mathbb{P}(\Gamma_{\varepsilon,R,K} \cap \{|M_k^\varepsilon| < \varepsilon \pm 2\varepsilon^2 \text{ and } \forall \ell = k+1, \ldots, n : |M_\ell^\varepsilon| \ge \varepsilon \pm 2\varepsilon^2\})$ for $1 \le k \le n$, and $p_0^\pm = p_{0,n,\varepsilon,R,K}^\pm := \mathbb{P}(\Gamma_{\varepsilon,R,K} \cap \{\forall \ell = 1, \ldots, n : |M_\ell^\varepsilon| \ge \varepsilon \pm 2\varepsilon^2\})$. The following estimates are the basis of the argument to follow.

Lemma 1. *For arbitrary R, K, and $0 < \gamma' < 2f_X(0) < \gamma''$, there is some ε_1 such that for $0 < \varepsilon < \varepsilon_1$ and $n_\varepsilon > m_\varepsilon \ge (\log \varepsilon)^4$,*

$$\begin{aligned}\mathbb{P}(\Gamma_{\varepsilon,R,K}) \ge \mathbb{P}(\tau_\varepsilon > n_\varepsilon) &+ \mathbb{P}(\Gamma_{\varepsilon,R,K})\gamma'\varepsilon \sum_{k=m_\varepsilon}^{n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k)}{A_k} \\ &- \mathbb{P}(\Gamma_{\varepsilon,R,K}) 8\gamma'\varepsilon^3 (G(n_\varepsilon) - G(m_\varepsilon)) - \mathbb{P}(\bigcup_{i=1}^{K} \{|M_i^\varepsilon| > R\}),\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(\Gamma_{\varepsilon,R,K}) \le \mathbb{P}(\tau_\varepsilon > n_\varepsilon) &+ \mathbb{P}(\Gamma_{\varepsilon,R,K})\gamma''\varepsilon \sum_{k=m_\varepsilon}^{n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k)}{A_k} + \mathbb{P}(\Gamma_{\varepsilon,R,K} \setminus \Omega^*) \\ &+ \mathbb{P}(\Gamma_{\varepsilon,R,K}) 8\gamma''\varepsilon^3 (G(n_\varepsilon) - G(m_\varepsilon)) + \mathbb{P}(\Omega^* \cap \{\tau_{3\varepsilon} \le m_\varepsilon\}).\end{aligned}$$

Proof. For the course of this proof, we simplify notations by suppressing the parameters ε, R, and K in m_ε, n_ε, M_i^ε, and $\Gamma_{\varepsilon,R,K}$. We will apply (8) with $n = n_\varepsilon$. Also, let $\nu := \varepsilon^2$.

(i) Starting with the $k = 0$ term, we see that

$$p_0^- \ge \mathbb{P}(\Gamma \cap \{\forall \ell = 1, \ldots, n : |M_\ell| \ge \varepsilon\}) \ge \mathbb{P}(\Gamma \cap \{\tau_\varepsilon > n\}).$$

We now consider the case where $m \le k \le n$. Let $\mathcal{A} := (2\nu\mathbb{Z}) \cap (-\varepsilon + 3\nu, \varepsilon - 3\nu)$. Notice that the sets $Q_a := (a - \nu, a + \nu)$ with $a \in \mathcal{A}$ are disjoint and contained in $(-\varepsilon + 2\nu, \varepsilon - 2\nu)$. Therefore the kth term in equation (8) satisfies

$$\begin{aligned} p_k^- &\ge \sum_{a \in \mathcal{A}} \mathbb{P}(\Gamma \cap \{M_k \in Q_a \text{ and } \forall \ell = k+1, \ldots, n : \ |M_\ell| \ge \varepsilon - 2\nu\}) \\ &\ge \sum_{a \in \mathcal{A}} \mathbb{P}(\Gamma \cap \{M_k \in Q_a \text{ and } \forall \ell = k+1, \ldots, n : |S_\ell - S_k + a| \ge \varepsilon - \nu\}) \\ &= \sum_{a \in \mathcal{A}} \mathbb{P}(\Gamma \cap \{M_k \in Q_a\})\, \mathbb{P}(\forall \ell = 1, \ldots, n - k : |S_\ell + a| \ge \varepsilon - \nu) \end{aligned} \tag{9}$$

by independence (where we assume that ε is so small that $(\log \varepsilon)^4 > K$). Note that

$$\mathbb{P}(\Gamma \cap \{M_k \in Q_a\}) = \int_{\{\forall i: x_i \ne x_0, |x_i| \le R\}} \mathbb{P}(S_{k-K} \in Q_a - x_K)\, d\mathbb{P}_{(M_0,\ldots,M_K)}(x_0, \ldots, x_K),$$

with $d\mathbb{P}_{(M_0,\ldots,M_K)}$ denoting the distribution of $(M_0, \ldots, M_K)$. Now fix θ as in Proposition 1, and $c \in (0, 1)$ such that $\gamma' < 2f_X(0)/c$. Elementary considerations (based on our condition on $m = m_\varepsilon$) show that Proposition 1 applies to $I = \frac{1}{A_{k-K}}(Q_a - x_K)$ if ε is sufficiently small, and in this case gives

$$\mathbb{P}(\Gamma \cap \{M_k \in Q_a\}) \ge \mathbb{P}(\Gamma) \frac{\gamma'\nu}{A_k}. \tag{10}$$

Using this, plus the observation that conditioning on $\{M_0 \in Q_a\}$ amounts to looking at $M_n^* := M_0^* + S_n$, $n \geq 0$, with M_0^* uniformly distributed on Q_a, we can continue to estimate, for small ε,

$$\begin{aligned} p_k^- &\geq \mathbb{P}(\Gamma) \frac{\gamma' \nu}{A_k} \sum_{a \in \mathcal{A}} \mathbb{P}(\forall \ell = 1, \dots, n-k : |S_\ell + a| \geq \varepsilon - \nu\}) \\ &\geq \mathbb{P}(\Gamma) \frac{\gamma' \nu}{A_k} \sum_{a \in \mathcal{A}} \mathbb{P}(\{\forall \ell = 1, \dots, n-k : |M_\ell| \geq \varepsilon\} \mid \{M_0 \in Q_a\}) \\ &\geq \mathbb{P}(\Gamma) \frac{\gamma' \varepsilon}{A_k} \sum_{a \in \mathcal{A}} \mathbb{P}(\{\forall \ell = 1, \dots, n-k : |M_\ell| \geq \varepsilon\} \cap \{M_0 \in Q_a\}) \\ &\geq \mathbb{P}(\Gamma) \frac{\gamma' \varepsilon}{A_k} \left(\mathbb{P}(\forall \ell = 1, \dots, n-k : |M_\ell| \geq \varepsilon) - \mathbb{P}(\varepsilon - 4\nu \leq |M_0| \leq \varepsilon)\right) \\ &= \mathbb{P}(\Gamma) \frac{\gamma' \varepsilon}{A_k} \left(\mathbb{P}(\tau_\varepsilon > n-k) - 8\nu\right). \end{aligned} \tag{11}$$

Putting together these estimates via Equation (8) gives

$$\mathbb{P}(\Gamma \cap \{\tau_\varepsilon > n\}) + \mathbb{P}(\Gamma)\gamma'\varepsilon \sum_{k=m}^{n} \frac{\mathbb{P}(\tau_\varepsilon > n-k)}{A_k} \leq \mathbb{P}(\Gamma) + \mathbb{P}(\Gamma) 8\gamma'\varepsilon\nu(G(n) - G(m)).$$

Since $\Gamma^c \cap \{\tau_\varepsilon > n\} \subseteq \bigcup_{i=1}^{K} \{|M_i| > R\}$ for ε so small that $n = n_\varepsilon > K$, this proves the first assertion of the lemma.

(ii) We only provide a sketch of the proof of the second point since the arguments are very similar to the above. Using Equation (8) gives

$$\mathbb{P}(\Gamma) \leq \mathbb{P}(\Gamma \cap \{\tau_\varepsilon > n\}) + \mathbb{P}(\Gamma \setminus \Omega^*) + \mathbb{P}(\Omega^* \cap \{\tau_{3\varepsilon} \leq m\}) + \sum_{k=m}^{n} p_k^+,$$

since $\sum_{k=1}^{m} p_k^+ \leq \mathbb{P}(\Gamma \cap \{\tau_{3\varepsilon} \leq m\})$. Next, take $\bar{\mathcal{A}} := (2\nu\mathbb{Z}) \cap (-\varepsilon - 3\nu, \varepsilon + 3\nu)$ and intervals $\bar{Q}_a := [a - \nu, a + \nu]$, $a \in \bar{\mathcal{A}}$, which cover $(-\varepsilon - 2\nu, \varepsilon + 2\nu)$. We can then use arguments parallel to those of part (i) to obtain

$$\begin{aligned} \sum_{k=m}^{n} p_k^+ &\leq \sum_{k=m}^{n} \sum_{a \in \bar{\mathcal{A}}} \mathbb{P}(\Gamma \cap \{M_k \in \bar{Q}_a \text{ and } \forall \ell = k+1, ..., n : \ |M_\ell| > \varepsilon + 2\nu)\}) \\ &\ \ \vdots \\ &\leq \mathbb{P}(\Gamma)\gamma''\varepsilon \sum_{k=m}^{n} \frac{\mathbb{P}(\tau_\varepsilon > n-k)}{A_k} + \mathbb{P}(\Gamma) 8\gamma''\varepsilon\nu(G(n) - G(m)), \end{aligned}$$

which proves our claim. □

Suitable choice of the n_ε then enables us to derive an asymptotic bound for the tails of the distributions of the $\varepsilon G(\tau_\varepsilon)$ as $\varepsilon \to 0$.

Lemma 2. *For all $\alpha \in [1, 2]$ and any $t > 0$ we have*

$$\limsup_{\varepsilon \to 0} \mathbb{P}(\gamma\varepsilon G(\tau_\varepsilon) > t) \leq \frac{\mathbb{P}(\Omega^*)}{1+t}.$$

Proof. Fix t, R, K, and $0 < \gamma' < 2f_X(0)$. For $\varepsilon > 0$ we take $m_\varepsilon := (\log \varepsilon)^4$ and choose n_ε so that $G(n_\varepsilon) \leq \frac{t}{\gamma\varepsilon} \leq G(n_\varepsilon + 1)$, whence $\mathbb{P}(\varepsilon\gamma G(\tau_\varepsilon) > t) \sim \mathbb{P}(\tau_\varepsilon > n_\varepsilon)$. As in the proof of Theorem

1 we see that $G(m_\varepsilon) = o(G(n_\varepsilon))$. Therefore

$$\varepsilon \sum_{k=m_\varepsilon}^{n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k)}{A_k} \geq \varepsilon(G(n_\varepsilon) - G(m_\varepsilon))\mathbb{P}(\tau_\varepsilon > n_\varepsilon) \sim \frac{t}{\gamma}\, \mathbb{P}(\tau_\varepsilon > n_\varepsilon). \tag{12}$$

Together with the first part of Lemma 1, this yields

$$\limsup_{\varepsilon\to 0} \mathbb{P}(\varepsilon\gamma G(\tau_\varepsilon) > t) \leq \frac{\mathbb{P}(\Gamma_{R,K}) + \mathbb{P}(\exists 1 \leq i \leq K : |S_i| > R-1)}{1 + t\frac{\gamma'}{\gamma}\mathbb{P}(\Gamma_{R,K})},$$

since $8\gamma'\varepsilon^3(G(n_\varepsilon) - G(m_\varepsilon)) \to 0$.

Taking successively $R \to \infty$, then $K \to \infty$, and finally $\gamma' \to 2f_X(0)$, we obtain the lemma. □

When $\alpha = 1$, this upper bound actually is the limit:

Lemma 3. *If $\alpha = 1$, then for any $t > 0$ we have*

$$\liminf_{\varepsilon\to 0} \mathbb{P}(\gamma\varepsilon G(\tau_\varepsilon) > t) \geq \frac{\mathbb{P}(\Omega^*)}{1+t}.$$

Proof. Fix t, R, K, and $\gamma'' > 2f_X(0)$, and choose m_ε and n_ε as in the previous proof. Similar to that situation we have $\mathbb{P}(\Gamma_{\varepsilon,R,K})8\gamma''\varepsilon^3(G(n_\varepsilon) - G(m_\varepsilon)) \to 0$, and, as a consequence of Theorem 1, also $\mathbb{P}(\Omega^* \cap \{\tau_{3\varepsilon} \leq m_\varepsilon\}) \to 0$.

Since $\alpha = 1$ means that G is slowly varying, we have $G(2n_\varepsilon) - G(n_\varepsilon) = o(G(n_\varepsilon))$. Hence

$$\begin{aligned}
&\mathbb{P}(\tau_\varepsilon > 2n_\varepsilon) + \mathbb{P}(\Gamma_{\varepsilon,R,K})\gamma''\varepsilon \sum_{k=m_\varepsilon}^{2n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > 2n_\varepsilon - k)}{A_k} \\
&\leq \mathbb{P}(\tau_\varepsilon > n_\varepsilon) + \mathbb{P}(\Gamma_{\varepsilon,R,K})\gamma''\varepsilon \left(\sum_{k=m_\varepsilon}^{n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon)}{A_k} + \sum_{k=n_\varepsilon}^{2n_\varepsilon} \frac{1}{A_k} \right) \\
&\leq \mathbb{P}(\tau_\varepsilon > n_\varepsilon) + \mathbb{P}(\Gamma_{\varepsilon,R,K})\gamma''\varepsilon G(n_\varepsilon)[\mathbb{P}(\tau_\varepsilon > n_\varepsilon) + o(1)] \\
&\leq \mathbb{P}(\tau_\varepsilon > n_\varepsilon) + t\frac{\gamma''}{\gamma}\mathbb{P}(\Gamma_{\varepsilon,R,K})\mathbb{P}(\tau_\varepsilon > n_\varepsilon) + o(1).
\end{aligned} \tag{13}$$

Combining these observations with the second estimate of Lemma 1 (replacing n_ε by $2n_\varepsilon$) entails

$$\liminf_{\varepsilon\to 0} \mathbb{P}(\tau_\varepsilon > n_\varepsilon) \geq \frac{\mathbb{P}(\Gamma_{R,K}) - \mathbb{P}(\Gamma_{R,K} \setminus \Omega^*)}{1 + t\frac{\gamma''}{\gamma}\mathbb{P}(\Gamma_{R,K})}.$$

We conclude by successively taking $R \to \infty$, $K \to \infty$, and $\gamma'' \to 2f_X(0)$. □

Proof of Theorem 4. Immediate from Lemmas 2 and 3, as $\varepsilon G(\tau_\varepsilon) \to 0$ outside Ω^*. □

When $\alpha \in (1, 2]$, Lemma 1 does not yet give the limit distribution. Still, it immediately implies the tightness of the family of distributions with the normalisation given there:

Lemma 4. *The family of distributions of the random variables $\varepsilon G(\tau_\varepsilon)$, $\varepsilon \in (0,1)$, is tight.*

Hence it will be enough to prove that the advertised limit law is the only possible accumulation point of our distributions. We henceforth abbreviate

$$Z_\varepsilon := \frac{\gamma}{\beta}\, \varepsilon G(\tau_\varepsilon), \quad \varepsilon > 0.$$

Lemma 5. *Suppose that $\alpha \in (1,2]$. Let $(\varepsilon_p)_{p\geq 1}$ be a positive sequence with $\lim_{p\to\infty}\varepsilon_p = 0$, and such that the conditional distributions of the Z_{ε_p} on Ω^* converge to the law of some random variable Y. Then its tail satisfies the integral equation*

$$1 = \Pr(Y>t) + t\int_0^1 \frac{\Pr(Y > t(1-u)^{\frac{1}{\beta}})}{u^{\frac{1}{\alpha}}}\,du \quad \forall t>0.$$

Proof. *(i)* We write $f(t) := \Pr(Y>t)$, and first prove that

$$\forall t>0,\quad 1 \geq f(t) + t\int_0^1 u^{-\frac{1}{\alpha}} f(t(1-u)^{\frac{1}{\beta}})\,du.$$

Let us only consider ε belonging to $\{\varepsilon_p,\ p\geq 1\}$. Note that by monotonicity and right continuity of f it suffices to prove the inequality for all $t\in(0,\infty)$ such that, for all $N\geq 1$ and all $r=0,...,N-1$, the function f is continuous at $t\left(1-\frac{r}{N}\right)^{\frac{1}{\beta}}$. Henceforth such a t will be fixed.

Now take some $\delta>0$, and choose $N_\delta > 1$ such that for all $N\geq N_\delta$,

$$\left|\int_0^1 \frac{f(t(1-u)^{\frac{1}{\beta}})}{u^{\frac{1}{\alpha}}}\,du - \frac{1}{N}\sum_{r=1}^{N-1}\frac{f(t(1-(r/N))^{\frac{1}{\beta}})}{((r+1)/N)^{\frac{1}{\alpha}}}\right| \leq \delta.$$

Now fix integers $N\geq N_\delta$, $K\geq 1$, and some $0<\gamma'<2f_X(0)$. For $\varepsilon>0$ small enough take n_ε such that $G(n_\varepsilon)\leq \frac{\beta t}{\gamma\varepsilon} < G(n_\varepsilon+1)$ (and hence $G(n_\varepsilon)\sim\frac{\beta t}{\gamma\varepsilon}$). Finally, let $m_\varepsilon := n_\varepsilon/N$.

According to the first point of Lemma 1, we have

$$\mathbb{P}(\Gamma_{\varepsilon,R,K}) \geq \mathbb{P}(Z_\varepsilon > t) + \mathbb{P}(\Gamma_{\varepsilon,R,K})\gamma'\varepsilon\sum_{k=m_\varepsilon}^{n_\varepsilon}\frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k)}{A_k}$$
$$- \mathbb{P}(\Gamma_{\varepsilon,R,K})8\gamma'\varepsilon^3(G(n_\varepsilon)-G(m_\varepsilon)) - \mathbb{P}(\bigcup_{i=1}^K\{|M_i^\varepsilon|>R\}),$$

Due to our assumption on the Z_{ε_p} and t, we see that $\mathbb{P}(Z_\varepsilon > t)\to\mathbb{P}(\Omega^*)f(t)$ as $\varepsilon_p\to 0$. Next, by monotonicity,

$$\sum_{k=n_\varepsilon/N}^{n_\varepsilon}\frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k)}{A_k} \geq \sum_{r=1}^{N-1}\sum_{k=0}^{n_\varepsilon/N-1}\frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k - (rn_\varepsilon/N))}{A_{k+(rn_\varepsilon/N)}}$$
$$\geq \sum_{r=1}^{N-1}\left(G\left(\frac{r+1}{N}\,n_\varepsilon\right) - G\left(\frac{r}{N}\,n_\varepsilon\right)\right)\mathbb{P}\left(\tau_\varepsilon > \left(1-\frac{r}{N}\right)n_\varepsilon\right).$$

By regular variation, the first term of the product is asymptotically equivalent to

$$G(n_\varepsilon)\left[\left(\frac{r+1}{N}\right)^{\frac{1}{\beta}} - \left(\frac{r}{N}\right)^{\frac{1}{\beta}}\right] \geq \frac{G(n_\varepsilon)}{\beta N\left(\frac{r+1}{N}\right)^{\frac{1}{\alpha}}}$$

as $\varepsilon_p\to 0$. On the other hand, the second term is equal to

$$\mathbb{P}\left(Z_\varepsilon > \varepsilon\frac{\gamma}{\beta}G\left(\left(1-\frac{r}{N}\right)n_\varepsilon\right)\right)\to\mathbb{P}(\Omega^*)\ f\left(t\left(1-\frac{r}{N}\right)^{\frac{1}{\beta}}\right),$$

since $G\left(\left(1-\frac{r}{N}\right)n_\varepsilon\right) \sim \left(1-\frac{r}{N}\right)^{\frac{1}{\beta}} G(n_\varepsilon)$. As a consequence, we see that

$$\begin{aligned}\liminf_{p\to\infty} \varepsilon_p \sum_{k=n_{\varepsilon_p}/N}^{n_{\varepsilon_p}} \frac{\mathbb{P}(\tau_{\varepsilon_p} > n_{\varepsilon_p} - k)}{A_k} &\geq \mathbb{P}(\Omega^*)\frac{t}{\gamma}\frac{1}{N}\sum_{r=1}^{N-1} \frac{f(t\left(1-\frac{r}{N}\right)^{1-\frac{1}{\alpha}})}{\left(\frac{r+1}{N}\right)^{\frac{1}{\alpha}}} \\ &\geq \mathbb{P}(\Omega^*)\frac{t}{\gamma}\left(\int_0^1 \frac{f(t(1-u)^{1-\frac{1}{\alpha}})}{u^{\frac{1}{\alpha}}}\,du - \delta\right).\end{aligned} \tag{14}$$

Furthermore, we again have $\mathbb{P}(\Gamma_{\varepsilon,R,K})8\gamma'\varepsilon^3(G(n_\varepsilon) - G(m_\varepsilon)) \to 0$. Moreover, $\mathbb{P}(\bigcup_{i=1}^K \{|M_i^\varepsilon| > R\}) \to \mathbb{P}(\bigcup_{i=1}^K \{|S_i| > R\})$. Combining all these asymptotic estimates and taking the limit $\varepsilon_p \to 0$, we end then up with

$$\mathbb{P}(\Gamma_{R,K}) \geq \mathbb{P}(\Omega^*)\left[f(t) + \frac{\mathbb{P}(\Gamma_{R,K})\gamma' t}{\gamma}\left(\int_0^1 \frac{f(t(1-u)^{1-\frac{1}{\alpha}})}{u^{\frac{1}{\alpha}}}\,du - \delta\right)\right] - \mathbb{P}(\bigcup_{i=1}^K \{|S_i| > R\}).$$

Successively letting $R \to \infty$, $K \to \infty$, $\gamma' \to 2f_X(0)$ and $\delta \to 0$ we obtain the desired inequality.

(ii) The converse inequality is proved analogously, using the other half of Lemma 1 and the fact that $\mathbb{P}(\Omega^* \cap \{\tau_{3\varepsilon} \leq m_\varepsilon\}) = o(1)$. □

Now let us identify the limit distribution satisfying the equality given by Lemma 5. To this end we consider the variables

$$Z'_\varepsilon := \left(\frac{\gamma}{\beta}\right)^\beta \frac{\tau_\varepsilon}{G^{-1}(1/\varepsilon)}, \quad \varepsilon > 0.$$

Lemma 6. *The conditional distributions of the Z_{ε_p} converge to a random variable Y iff the conditional distributions of the Z'_{ε_p} converge to Y^β. The latter then satisfies*

$$1 = \Pr(Y^\beta > t) + \int_0^t \frac{\Pr(Y^\beta > t - v)}{v^{\frac{1}{\alpha}}}\,dv \quad \forall t > 0.$$

Proof. The equivalence of the two conditional distributional convergence statements follows from regular variation of G^{-1}, see e.g. Lemma 1 of [4]. Suppose that they hold. Then, according to Lemma 5, for any $t > 0$, we have

$$1 = \Pr(Y^\beta > t) + t^{\frac{1}{\beta}} \int_0^1 \frac{\Pr(Y^\beta > t(1-u))}{u^{\frac{1}{\alpha}}}\,du,$$

and the conclusion follows by a change of variables, $v = tu$. □

Lemma 7. *Let W be a random variable with values in $[0, \infty)$ satisfying*

$$\Pr(W \leq t) = \int_0^t \frac{\Pr(W > t - v)}{v^{\frac{1}{\alpha}}}\,dv \quad \forall t > 0. \tag{15}$$

Then

$$\mathbb{E}\left[e^{-sW}\right] = \frac{1}{1 + c_\beta s^{\frac{1}{\beta}}} \quad \forall s > 0,$$

with $c_\beta := \Gamma(\frac{1}{\beta})^{-1}$. In particular, the distribution of W coincides with that of $c_\beta^\beta \mathcal{E}^\beta \mathcal{G}_{\frac{1}{\beta}}$, where the independent variables $\mathcal{E}$ and $\mathcal{G}_{\frac{1}{\beta}}$ are as in the statement of Theorem 3.

Proof. Let $s > 0$. We have

$$\mathbb{E}[e^{-sW}] = \int_0^{+\infty} \Pr(e^{-sW} \geq u)\, du = \int_0^{+\infty} \Pr\left(W \leq -\frac{\log(u)}{s}\right) du = \int_0^{+\infty} \Pr\left(W \leq v\right) se^{-sv}\, dv.$$

Hence, for any $s > 0$, we find

$$\begin{aligned}
\mathbb{E}[e^{-sW}] &= \int_0^{+\infty} \left[\int_0^v \frac{\Pr(W \geq v-w)}{w^{\frac{1}{\alpha}}}\, dw\right] se^{-sv}\, dv \\
&= \int_0^{+\infty} \frac{1}{w^{\frac{1}{\alpha}}} \left[\int_w^{+\infty} \Pr(W \geq v-w) se^{-sv}\, dv\right] dw \\
&= \int_0^{+\infty} \frac{e^{-sw}}{w^{\frac{1}{\alpha}}} \left[\int_0^{+\infty} \Pr(W \geq z) se^{-sz}\, dz\right] dw \\
&= \int_0^{+\infty} \frac{e^{-sw}}{w^{\frac{1}{\alpha}}} \left[1 - \int_0^{+\infty} \Pr(W \leq z) se^{-sz}\, dz\right] dw \\
&= \int_0^{+\infty} \frac{e^{-sw}}{w^{\frac{1}{\alpha}}}\, dw \cdot \left[1 - \mathbb{E}[e^{-sW}]\right],
\end{aligned}$$

and our claim about the Laplace transform of W follows since

$$\int_0^{+\infty} \frac{e^{-sw}}{w^{\frac{1}{\alpha}}}\, dw = \frac{\beta}{s^{\frac{1}{\beta}}} \int_0^{+\infty} e^{-z^\beta}\, dz = \frac{1}{c_\beta s^{\frac{1}{\beta}}} \quad \text{with} \quad c_\beta := \frac{1}{\Gamma(\frac{1}{\beta})}.$$

Given this, a routine calculation (cf. XIII.11.10 of [9]) shows that W indeed has the same Laplace transform as $c_\beta^\beta \mathcal{E}^\beta \mathcal{G}_{\frac{1}{\beta}}$. □

Proof of Theorem 5. According to Lemma 4 the family of distributions of the Z_ε, $\varepsilon \in (0,1)$, is tight. By Lemma 5, Lemma 6 and Lemma 7, the law of $c_\beta\, \mathcal{E}\mathcal{G}_{1/\beta}^{1/\beta}$ is the only possible accumulation point of these distributions. □

4. Convergence in distribution for $\mathbf{T}_\varepsilon^x$

To complete the proof of Theorems 2 and 3 we now utilize Theorems 4 and 5. Note first that it suffices to prove Theorems 2 and 3 under the additional assumption that $S_0' = 0$, in which case

$$\mathbf{T}_\varepsilon^x = \hat{\mathbf{T}}_\varepsilon^x := \inf\{n \geq 1 : |S_n - x| < \varepsilon\} \quad \text{and} \quad \Omega_x^* = \hat{\Omega}_x^* := \{S_n \neq x\ \forall n\}.$$

Indeed, in the situation of Theorem 2, with arbitrary distribution P of S_0', we then have

$$\mathbb{P}\left(\gamma\varepsilon\, G(\mathbf{T}_\varepsilon^x) \leq t\right) = \int_{\mathbb{R}} \mathbb{P}\left(\gamma\varepsilon\, G(\hat{\mathbf{T}}_\varepsilon^{x-y}) \leq t\right) dP(y) \to \int_{\mathbb{R}} \mathbb{P}\left(\hat{\Omega}_{x-y}^*\right) dP(y) \cdot \frac{t}{1+t}$$

by the $P = \delta_0$ case of Theorem 2 and dominated convergence. Analogously for Theorem 3.

Therefore, for the remainder of this section we assume that $S_0' = 0$.

Next, we observe that our key lemma (Lemma 1) can be adapted as follows. Let $\Gamma_{R,K}^x$ be the event defined by

$$\Gamma_{R,K}^x := \{\forall i = 1, \ldots, K : S_i \neq x \text{ and } |S_i| \leq R\}.$$

Lemma 8. *For arbitrary R, K, and $0 < \gamma' < 2f_X(0) < \gamma''$, there is some $\varepsilon_1 > 0$ such that for $0 < \varepsilon < \varepsilon_1$ and $n_\varepsilon > m_\varepsilon \geq (\log \varepsilon)^4$,*

$$\mathbb{P}(\Gamma^x_{R,K}) \geq \mathbb{P}(\mathbf{T}^x_\varepsilon > n_\varepsilon) + \mathbb{P}(\Gamma^x_{R,K})\gamma'\varepsilon \sum_{k=m_\varepsilon}^{n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k)}{A_k} - \mathbb{P}(\Gamma^x_{R,K}) 8\gamma'\varepsilon^3 (G(n_\varepsilon) - G(m_\varepsilon)) - \mathbb{P}(\bigcup_{i=1}^{K} \{|S_i| > R\}),$$

and

$$\mathbb{P}(\Gamma^x_{R,K}) \leq \mathbb{P}(\mathbf{T}^x_\varepsilon > n_\varepsilon) + \mathbb{P}(\Gamma^x_{R,K})\gamma''\varepsilon \sum_{k=m_\varepsilon}^{n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k)}{A_k} + \mathbb{P}(\Gamma^x_{R,K} \setminus \Omega^*_x) + \mathbb{P}(\Gamma^x_{R,K}) 8\gamma''\varepsilon^3 (G(n_\varepsilon) - G(m_\varepsilon)) + \mathbb{P}(\Omega^*_x \cap \{\mathbf{T}_{3\varepsilon} \leq m_\varepsilon\}).$$

Proof of Lemma 8. We have the following analogue of formula (8),

$$\mathbb{P}(\Gamma^x_{R,K}) = \sum_{k=0}^{n_\varepsilon} p_k^{x,-} = \sum_{k=0}^{n_\varepsilon} p_k^{x,+}, \tag{16}$$

with

$$p_0^{x,\pm} := \mathbb{P}(\Gamma^x_{R,K} \cap \{\forall \ell = 1, \ldots, n_\varepsilon : |S_\ell - x| \geq \varepsilon \pm 2\varepsilon^2\})$$

and

$$p_k^{x,\pm} := \mathbb{P}(\Gamma^x_{R,K} \cap \{|S_k - x| < \varepsilon \pm 2\varepsilon^2 \text{ and } \forall \ell = k+1, \ldots, n_\varepsilon : |S_\ell - x| \geq \varepsilon \pm 2\varepsilon^2\}).$$

We follow the proof of Lemma 1.

(i) Observe first that

$$p_0^{x,-} \geq \mathbb{P}(\Gamma^x_{R,K} \cap \{\mathbf{T}^x_\varepsilon > n_\varepsilon\}).$$

Now consider indices with $m_\varepsilon \leq k \leq n_\varepsilon$. With the same set $\mathcal{A}$ as in the proof of Lemma 1, we find, arguing as in (9), that

$$\begin{aligned} p_k^{x,-} &\geq \sum_{a \in \mathcal{A}} \mathbb{P}(\Gamma^x_{R,K} \cap \{S_k - x \in Q_a \text{ and } \forall \ell = k+1, \ldots, n_\varepsilon : \ |S_\ell - x| \geq \varepsilon - 2\nu\}) \\ &\geq \sum_{a \in \mathcal{A}} \mathbb{P}(\Gamma^x_{R,K} \cap \{S_k - x \in Q_a\}) \ \mathbb{P}(\forall \ell = 1, \ldots, n_\varepsilon - k : |S_\ell + a| \geq \varepsilon - \nu). \end{aligned}$$

A proof parallel to that of (10) shows that

$$\mathbb{P}(\Gamma^x_{R,K} \cap \{S_k - x \in Q_a\}) \geq \mathbb{P}(\Gamma^x_{R,K}) \frac{\gamma'\nu}{A_k}$$

if ε is sufficiently small. Therefore,

$$\begin{aligned} p_k^{x,-} &\geq \mathbb{P}(\Gamma^x_{R,K}) \frac{\gamma'\nu}{A_k} \sum_{a \in \mathcal{A}} \mathbb{P}(\forall \ell = 1, \ldots, n_\varepsilon - k : |S_\ell + a| \geq \varepsilon - \nu\}) \\ &\geq \mathbb{P}(\Gamma^x_{R,K}) \frac{\gamma'\varepsilon}{A_k} \left(\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k) - 8\nu \right), \end{aligned}$$

where the second step uses an estimate contained in (11). Continuing as in the proof of Lemma 1, we obtain the first assertion of our lemma.

(ii) Similar adaptations give the second assertion of the lemma. □

We can now complete the proofs of our main distributional limit theorems:

Proof of Theorem 2. We go back to Lemmas 2 and 3, observing that we already have (6) at our disposal. Take $t \in (0,\infty)$, $R, K \geq 1$, and $\gamma' < 2f_X(0) < \gamma''$. For $\varepsilon > 0$ let $m_\varepsilon := (\log\varepsilon)^4$ and choose n_ε, such that $G(n_\varepsilon) \leq \frac{t}{\gamma\varepsilon} \leq G(n_\varepsilon + 1)$, meaning that $\mathbb{P}(\varepsilon\gamma G(\mathbf{T}_\varepsilon^x) > t) \sim \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon)$.

In view of (6), the estimate (12) of Lemma 2 becomes

$$\liminf_{\varepsilon\to 0} \varepsilon \sum_{k=m_\varepsilon}^{n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k)}{A_k} \geq \frac{\mathbb{P}(\Omega^*)}{\gamma} \frac{t}{1+t}.$$

Combining this with the first part of Lemma 8 leads to

$$\limsup_{\varepsilon\to 0} \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon) \leq \mathbb{P}(\Gamma_{R,K}^*) \left(1 - \frac{\gamma'}{2f_X(0)} \frac{t}{1+t}\right) + \mathbb{P}(\exists 1 \leq i \leq K : |S_i| > R - 1).$$

Successively letting $R \to \infty$, then $K \to \infty$, and finally $\gamma' \to 2f_X(0)$, we obtain

$$\limsup_{\varepsilon\to 0} \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon) \leq \frac{\mathbb{P}(\Omega_x^*)}{1+t}.$$

To get the corresponding lower bound, recall that $\mathbb{P}(\Omega^* \cap \{\mathbf{T}_{3\varepsilon}^x \leq m_\varepsilon\}) \to 0$ by Theorem 1. Parallel to (13) we have

$$\mathbb{P}(\mathbf{T}_\varepsilon^x > 2n_\varepsilon) + \mathbb{P}(\Gamma_{R,K}^x)\gamma''\varepsilon \sum_{k=m_\varepsilon}^{2n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > 2n_\varepsilon - k)}{A_k} \leq \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon) + t\frac{\gamma''}{\gamma}\mathbb{P}(\Gamma_{R,K}^x)\mathbb{P}(\tau_\varepsilon > n_\varepsilon) + o(1).$$

Together with the second part of Lemma 8 (with n_ε replaced by $2n_\varepsilon$) and (6), this implies

$$\liminf_{\varepsilon\to 0} \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon) \geq \frac{\mathbb{P}(\Omega_x^*)}{1+t},$$

completing the proof. □

Proof of Theorem 3. We fix $t \in (0,\infty)$, and choose n_ε such that $G(n_\varepsilon) \leq \frac{\beta t}{\gamma\varepsilon} < G(n_\varepsilon + 1)$.

According to the proof of Theorem 5 (see, in particular, (14) in Lemma 5), we know that for m_ε with $m_\varepsilon = o(n_\varepsilon)$,

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{k=m_\varepsilon}^{n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k)}{A_k} = \frac{\mathbb{P}(\Omega^*)}{\gamma} \Pr(Y \geq t) =: \psi,$$

where $Y = \Gamma(\frac{1}{\beta})^{-1}\mathcal{E}\mathcal{G}_{1/\beta}^{1/\beta}$ is the limiting random variable of the $\gamma\beta^{-1}\varepsilon G(\tau_\varepsilon)$. Therefore, if we take $m_\varepsilon := (\log\varepsilon)^4$, then Lemma 8 implies that for $R, K \geq 1$ and $\gamma' < 2f_X(0) < \gamma''$,

$$\limsup_{\varepsilon\to 0} \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon) \leq \mathbb{P}(\Gamma_{R,K}^x)\left(1 - \gamma'\psi\right) + \mathbb{P}\left(\bigcup_{i=1}^K \{|S_i| > R\}\right)$$

and

$$\liminf_{\varepsilon\to 0} \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon) \geq \mathbb{P}(\Gamma_{R,K}^x)\left(1 - \gamma''\psi\right) - \mathbb{P}\left(\Gamma_{R,K} \setminus \Omega_x^*\right).$$

Since $\lim_{K\to+\infty}\lim_{R\to+\infty}\mathbb{P}(\Gamma_{R,K}^x) = \mathbb{P}(\Omega_x^*)$ and $\lim_{K\to+\infty}\lim_{R\to+\infty}\mathbb{P}\left(\bigcup_{i=1}^K\{|S_i| > R\}\right) = 0$, we get

$$\mathbb{P}(\Omega_x^*)\left(1 - \gamma''\psi\right) \leq \liminf_{\varepsilon\to 0} \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon) \leq \limsup_{\varepsilon\to 0} \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon) \leq \mathbb{P}(\Omega_x^*)\left(1 - \gamma'\psi\right),$$

and hence

$$\lim_{\varepsilon\to 0} \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon) = \mathbb{P}(\Omega_x^*)\left(1 - 2f_X(0)\psi\right) = \mathbb{P}(\Omega_x^*)\Pr(Y > t),$$

as required. □

Proof of Corollary 1. This is an $\alpha = 2$ case with $A_n = \sqrt{n}$ and $f_X(0) = \frac{1}{\sqrt{2\pi}}$. Recalling that $\mathcal{G}_{1/2} = \frac{1}{2\mathcal{N}^2}$ in distribution (cf. XIII.3.b of [9]) proves our claim. □

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