Geometry of Projective Complex Curves

Seiki Nishikawa

Mathematical Institute, Tohoku University, Sendai, Japan

Let M be a complex manifold of dimension n, and $T^{1,0}M \to M$ the holomorphic tangent bundle of M. We denote by $z = (z^i)$ and $(z, v) = (z^i, v^j)$ complex local coordinate systems in M and $T^{1,0}M$, respectively. Assume that M is equipped with a *strongly pseudoconvex complex Finsler metric* F, that is, a real-valued C^0 function $F : T^{1,0}M \to \boldsymbol{R}$ satisfying the following conditions:

(i) $F(z, v) \geq 0$, and $F(z, v) = 0$ if and only if $v = 0$.

(ii) $F(z, \lambda v) = |\lambda| F(z, v)$ for all $(z, v) \in T^{1,0}M$ and $\lambda \in \boldsymbol{C}$.

(iii) $F \in C^\infty(T^{1,0}M \setminus \{0\})$, that is, F is C^∞ outside the zero section of $T^{1,0}M$.

(iv) The complex Hessian

$$(G_{i\bar{j}}) = \left(\frac{\partial^2 G}{\partial v^i \partial \bar{v}^j} \right)$$

of $G = F^2$ is positive definite on $T^{1,0}M \setminus \{0\}$.

Let $\gamma : I \to M$ be a C^∞ curve defined on an open interval in $\boldsymbol{R}$. γ is called a *holomorphically planar curve* if it satisfies

$$\frac{d}{dt} \left(\frac{\partial G}{\partial \bar{v}^i}(\gamma(t), \dot{\gamma}(t)) \right) + \rho(\dot{\gamma}(t)) \frac{\partial G}{\partial \bar{v}^i}(\gamma(t), \dot{\gamma}(t)) = \frac{\partial G}{\partial \bar{z}^i}(\gamma(t), \dot{\gamma}(t)), \quad 1 \leq i \leq n$$

for some complex-valued 1-form ρ on M. By a *projective complex curve* of M we mean a holomorphic map $\varphi : \Delta \to M$ from the Poincaré disk Δ into M, which maps every geodesic σ in Δ parametrized by arc length to a holomorphically planar curve $\gamma = \varphi \circ \sigma$ in M.

The aim of this talk is to discuss the geometry of these projective complex curves. In particular, we study the following:

(i) The defining equation of projective complex curves.

(ii) The integrability condition of the defining equation of projective complex curves.

(iii) Relation of these curves to the projective flatness of complex Finsler metrics.

A special case of projective complex curves is a *complex geodesic* of M, which is a holomorphic map $\varphi : \Delta \to M$ mapping every geodesic σ in Δ parametrized by arc length to a geodesic $\gamma = \varphi \circ \sigma$ in M parametrized by arc length. The geometry of complex geodesics was studied, for instance, by E. Vesentini, and M. Abate and G. Patrizio, and applied to the classification problem of Kähler Finsler manifolds of constant holomorphic curvature.