Price dynamics in a two assets market game with asymmetric information

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Previous works

- "On the strategic Origin of Brownian Motion in Finance"
  B.De Meyer, H.Moussa Saley (2003)
  Int.J. of Game theory
- "Price Dynamics on a Stock Market with Asymmetric Information"
  B. De Meyer (2007)
  Cowles Foundation Discussion Paper

Plan

1. The model
   - The one asset game $G_n(\mu^A)$
   - The two assets game $\Gamma_n(\mu)$

2. Main Results
   - Value of the repeated game
   - Asymptotic results

The game $G_n(\mu^A)$

- Two players (P1 and P2) are trading a risky asset A against a numeraire N
- Liquidation value of the risky asset at date 1 : L^A

  The numéraire asset N has the constant value 1 over time.
- At date 0 : Nature selects L^A using a probability distribution μ^A.

  P1 is informed of L^A. P2 is not informed but knows μ^A
- n transaction rounds up to date 1.
- Action sets : I,J
- Trading device : $T(i,j) = (R(i,j), N(i,j))$
- P1's portfolio after round k : $y_k = (y_k^A, y_k^N)$ (z_k for P2)

  $$y_k = y_{k-1} + T(i_k, j_k)$$
  $$z_k = z_{k-1} - T(i_k, j_k)$$
- After each round, actions are publicly announced.

Payoff

- Strategy σ for P1 : sequence $(\sigma_1, ..., \sigma_n)$ of transition probabilities
$$\sigma_k : \mathbb{R} \times (I \times J)^{(k-1)} \rightarrow \Delta(I)$$
- Strategy τ for P2 :$(\tau_1, ..., \tau_n)$
$$\tau_k : (I \times J)^{(k-1)} \rightarrow \Delta(J)$$
- $(\mu^A, \sigma, \tau) \rightarrow \Pi_{(\mu^A, \sigma, \tau)} \in \Delta(\mathbb{R} \times I^n \times J^n)$
- P1 and P2 are risk neutral :
  P1's payoff : $\mathbb{E}[y_n^A L^A + y_n^N]$
  P2's payoff : $\mathbb{E}[z_n^A L^A + z_n^N]$
- We assume : $y_0 = 0$, $z_0 = 0$ so that $z_n = -y_n$
  $\Longrightarrow$ Aumann-Maschler zero-sum game with one-sided information

Price process

- Liquidation value of the risky asset A : L^A
- Price after round k := expected value of A for P2

  Information available : σ-field $\mathcal{F}_k = \sigma(i_1, j_1, ..., i_k, j_k)$

  Price process : $L^A_k = \mathbb{E}[L^A \mid \mathcal{F}_k]$
- With $\mathcal{F}_{n+1} = \sigma(i_1, j_1, ..., i_n, j_n, L^A)$ we have $L^A_{n+1} = L^A$ and the price process $(L^A_k)_{k=0,..,n+1}$ belongs to :

$$\mathcal{M}_n(\mu^A) = \{ \text{ martingales of length n+1 and terminal law } \mu^A \}$$

Hypothesis

We consider a general trading device T.

- We assume the following hypothesis (H) :

1) Existence of the value : $\forall \mu \in \Delta^2(\mathbb{R})$,$\forall n$, the game $G_n(\mu)$ has a value $v_n(\mu)$ and both players have optimal strategies.
2) Positive value of information : $\exists \mu \in \Delta^2(\mathbb{R})\ :\ v_1(\mu) > 0$.
3) L^p-continuity : There exists $p \in [1, 2[$ and $K \in \mathbb{R}$ such that for all r.v. $X \sim \mu$ and $Y \sim \nu$
$\mid v_1(\mu) - v_1(\nu) \mid \leq K \left\| X - Y \right\|_{L^p}$.
4) For all $c \in \mathbb{R}$ and X some random variable , $v_1[cX] = \mid c \mid v_1[X]$ and $v_1[X + c] = v_1[X]$.

Convergence of the value

- $\Delta^2(\mathbb{R})$: probabilities with finite variance
- For $\mu^A \in \Delta^2(\mathbb{R})$ and $Z \sim \mathcal{N}(0,1)$
  $\rightarrow f_{\mu^A} :=$ the unique right-continuous nondecreasing function such that $f_{\mu^A}(Z) \sim \mu^A$.

Theorem

$\forall \mu^A \in \Delta^2(\mathbb{R}),\ \frac{1}{\rho\sqrt{n}} v_n(\mu^A) \underset{n\to\infty}{\longrightarrow} \alpha(\mu^A)$
with $\rho = sup\{v_1[X] \mid \mathbb{E}[X] = 0; \|X\|_{L^2} \leq 1\}$ *and*

$$\begin{aligned}\alpha(\mu^A) &= max\{\mathbb{E}[L^A Z] \ \mid \ L^A \sim \mu^A \, and \, Z \sim \mathcal{N}(0,1)\} \\ &= \mathbb{E}[f_{\mu^A}(Z)Z]\end{aligned}$$

Asymptotic results

- If both players play an optimal strategy in $G_n(\mu^A)$
  $\rightarrow$ equilibrium price process :$((L_k^A)^n)_{k=0,\dots,n}$
- Π^n :continuous time representation of the price process
  $\forall t \in [0,1] \quad \Pi_t^n = L_{\lfloor nt \rfloor}^{An}$
- For B standard brownian motion and $\mathcal{G}_t$ its natural filtration
  $\rightarrow \Pi_t^{\mu^A} = \mathbb{E}[f_{\mu^A}(B_1) \mid \mathcal{G}_t]$: continuous martingale of maximal variation (CMMV) with terminal law μ^A.

Theorem

For $\mu^A \in \Delta^2(\mathbb{R})$, and under hypotheses (H), the process Π^n converges in finite-dimensional distributions to Π^{μ^A}.

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The two assets game $\Gamma_n(\mu)$

- two agents trading two risky assets A and B against a numeraire N during n transaction rounds up to date 1.
- date 0 : Nature selects $L = (L^A, L^B)$ with a probability distribution μ over $\mathbb{R}^2$.
- P1 is informed of L and P2 knows only μ.
- The same transaction device $T = (R, N)$ is used on both markets.

  Player's action sets : I^2 and J^2

  P1's portfolio after round k : $y_k = (y_k^A, y_k^B, y_k^N)$

  (i_k, j_k) : actions of the players at round k

  $y_k = y_{k-1} + \tilde{T}(i_k, j_k)$ with

$$\begin{aligned}\tilde{T}(i,j) &= (\quad R(i^A, j^A),\ R(i^B, j^B) \quad , \quad N(i^A, j^A) + N(i^B, j^B) \quad) \\ &= (\qquad \tilde{R}(i,j) \qquad , \qquad \tilde{N}(i,j) \qquad)\end{aligned}$$

Payoff function

- Strategy σ for P1 : sequence $(\sigma_1, ..., \sigma_n)$ of transition probabilities

$$\sigma_k : \mathbb{R}^2 \times (I^2 \times J^2)^{(k-1)} \to \Delta(I^2)$$

- Strategy τ for P2 :$(\tau_1, ..., \tau_n)$

$$\tau_k : (I^2 \times J^2)^{(k-1)} \to \Delta(I^2)$$

- $(\mu, \sigma, \tau) \to \Pi_{(\mu,\sigma,\tau)} \in \Delta(\mathbb{R}^2 \times I^{2n} \times J^{2n})$
- P1 and P2 are risk neutral and $y_0 = 0$, $z_0 = 0$ so that $z_n = -y_n$.
- P1's payoff :

$$\mathbb{E}[y_n^A L^A + y_n^B L^B + y_n^N]$$
$$= \mathbb{E}[< L. \sum_{k=1}^{n} \tilde{R}(i_k, j_k) > + \sum_{k=1}^{n} \tilde{N}(i_k, j_k)]$$

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Value of the one shot game $\Gamma_1(\mu)$

- $\Gamma_1(\mu)$: one round $\longrightarrow$ No incentive to conceal information.
- Payoff $=\mathbb{E}[R(i^A, j^A)L^A + N(i^A, j^A) + R(i^B, j^B)L^B + N(i^B, j^B)]$
- P1 and P2 can use optimal strategies of the one asset game for each asset.

Proposition

For $\mu \in \Delta^2(\mathbb{R}^2)$ with marginals (μ^A, μ^B), $\Gamma_1(\mu)$ has a value $V_1(\mu) = v_1(\mu^A) + v_1(\mu^B)$ and both players have optimal strategies.

Value of the n rounds game

The two assets A and B are linked
$\rightarrow$ Information on B contains information on A
P2 has more information than in two independent one asset games

$$V_n(\mu) \leq v_n(\mu^A) + v_n(\mu^B)$$

For a martingale $(L_k, \mathcal{F}_k)_{k=0,..,n+1} \in \mathcal{M}_n(\mu)$, we define the V_1-variation by :

$$\mathcal{V}_n^{V_1}(L_k, \mathcal{F}_k) = \mathbb{E}[\sum_{k=1}^{n} V_1([L_k^{(n)} \mid \mathcal{F}_{k-1}])].$$

Proposition (De Meyer 2007)

$V_n(\mu) = sup\{\mathcal{V}_n^{V_1}(L_k, \mathcal{F}_k) \mid (L_k, \mathcal{F}_k) \in \mathcal{M}_n(\mu)\}$
If both players play optimal strategies in $\Gamma_n(\mu)$, the the a posteriori martingale is optimal in the above supremum.

The same result holds for v_n with the v_1-variation.

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Convergence of the value

We assume (H) and :

(H') $\forall \mu \in \Delta^2(\mathbb{R}^2)$, $\Gamma_n(\mu)$ has a value and both players have optimal strategies.

Theorem

$$\forall \mu \in \Delta^2(\mathbb{R}^2),\ \frac{1}{\sqrt{n}} V_n(\mu) \underset{n\to\infty}{\longrightarrow} \rho V(\mu)$$

with $V(\mu) = sup \left\{ \mathbb{E}[L^A X_1 + L^B Y_1] \;\middle|\; L \sim \mu,\ X, Y \text{ bi-brownian} \right\}$
bi-brownian : process such that the two coordinates are brownian motions with respect to the same filtration.

Monotonic Derivative

We assume $\mu \in \Delta^2(\mathbb{R}^2)$ is such that $L^B = g(L^A)$ for some nondecreasing function g.
Let B be a standard brownian motion defined on $(\Omega, \mathcal{A}, \mathbb{P})$ and let $(\mathcal{G}_t)_{t\geq 0}$ denotes its natural filtration. Consider next the bivariate process $(\Pi^{\mu^A}, \Pi^{\mu^B})$ defined by :

$$\Pi_t^{\mu^A} = \mathbb{E}[f_{\mu^A}(B_1) \mid \mathcal{G}_t] \ , \ \Pi_t^{\mu^B} = \mathbb{E}[f_{\mu^B}(B_1) \mid \mathcal{G}_t]$$

with f_{μ^A} nondecreasing such that $f_{\mu^A}(B_1) \sim \mu^A$ (resp f_{μ^B}).

Theorem

If both players play an optimal strategy in $\Gamma^n(\mu)$, then the continuous versions of the a posteriori martingales $(L_k^{(n)}, \mathcal{F}_k^{(n)})_{k=0,..n+1}$ converge in finite-dimensional distributions to the process $(\Pi^{\mu^A}, \Pi^{\mu^B})$.

General case :The dual problem

Using a duality theorem from optimal transport theory

$$V(\mu) = sup \left\{ \mathbb{E}[L^A X_1 + L^B Y_1] \;\middle|\; L \sim \mu, (X, Y) \text{ bi-brownian} \right\}$$
$$= \inf_{\psi} \{ \mathbb{E}[\psi(L)] + \max_{bi-brownian} \mathbb{E}[\psi^*(X_1, Y_1)] \}$$

where ψ is a convex function and ψ^* its Fenchel transform.

$$V^*(\psi) = \max_{bi-brownian} \mathbb{E}[\psi^*(X_1, Y_1)]$$

For any bi-brownian process we have :

$$Y_t = \int_0^t c_s dX_s + \int_0^t \sqrt{1 - c_s^2} dZ_s$$

with Z a brownian motion independent of X and c_s the instantaneous correlation process ($\in [-1, 1]$).

The dual problem

The dual problem reduces to the stochastic control problem :

$$
\begin{aligned}
V^*(\psi) &= \max_{c_s \in [-1,1]} \mathbb{E}[\psi^*(X_1, \int_0^1 c_s dX_s + \int_0^1 \sqrt{1-c_s^2} dZ_s)] \\
&= \max_{c_s \in \{-1,1\}} \mathbb{E}[\psi^*(X_1, \int_0^1 c_s dX_s)]
\end{aligned}
$$

The value function of this problem :

$$
u(x,y,t) = \max_{c_s \in \{-1,1\}} \mathbb{E}[\psi^*(x + \int_t^1 dX_s, y + \int_t^1 c_s dX_s)]
$$

is a viscosity solution of the HJB equation :

$$
\begin{cases} -u_t = \frac{1}{2}\Delta u + \mid u_{xy} \mid \\ u(x,y,1) = \psi^*(x,y) \end{cases} \tag{1}
$$

Example of non-monotonic derivative

$\mu \in C = \{\}$ of probabilities over the three points :

$$I = (-1, 1); J = (0, 0); K = (1, 1)$$

- $L^B = \mid L^A \mid$ non monotonic derivative.
- dual variable $\psi^* \to \gamma^*(x + ., y + .)$ with $\gamma^*(u, v) = (v + |u|)^+$

$$u(x, y, t) = \max_{c_s \in \{-1,1\}} \mathbb{E}[\gamma^*(x + \int_t^1 dX_s, y + \int_t^1 c_s dX_s)]$$

- Using a geometric argument : $X_t, \int_0^t sgn(X_s) dX_s$
- $\overline{u}(x, y, t) = \mathbb{E}[\gamma^*(x + \int_t^1 dX_s, y + \int_t^1 sgn(X_s) dX_s)]$
  $\to C^2$-solution to (1)
- verification theorem from stochastic control theory
  $\to \overline{u}$: value function , $sgn(x)$: optimal control

Example of non-monotonic derivative

Proposition

$\forall \mu \in int(C)$, $\exists (x, y) \in \mathbb{R}^2$ such that :

$$L := \nabla \gamma^*(x + \int_0^1 dX_s, y + \int_0^1 sgn(X_s)dX_s) \sim \mu$$

$$V(\mu) = \mathbb{E}[L^A X_1 + L^B \int_0^1 sgn(X_s)dX_s]$$

(x, y) is the solution of :

$$(\frac{\partial \overline{u}}{\partial x}(x, y, 1), \frac{\partial \overline{u}}{\partial y}(x, y, 1)) = \mathbb{E}_\mu[L]$$

Example of trading mechanism

- Players post prices $i, j \in \mathbb{R}$ and one share of the risky asset A is transacted at the best price.
- $T^*(i,j) = \begin{cases} (\quad 1 \quad , \quad -i \quad) & \text{if} \quad i > j \\ (\quad -1 \quad , \quad j \quad) & \text{if} \quad i < j \\ (\quad 0 \quad , \quad 0 \quad) & \text{if} \quad i = j \end{cases}$
- Results for T^* :value of the one shot game (De Meyer , Moussa Saley 2003)

$$v_1(\mu^A) = max\{\mathbb{E}[L^A U^A] \mid L^A \sim \mu^A \, , \, U^A \sim \mathcal{U}_{[-1,1]}\}$$

Explicit solution : L^A nondecreasing function of U^A

Application to T^*

$$\begin{aligned} V_1(\mu) &= max\{\mathbb{E}[L^A U^A] \mid L^A \sim \mu^A\,,\ U^A \sim \mathcal{U}_{[-1,1]}\} \\ &\quad + max\{\mathbb{E}[L^B U^B] \mid L^B \sim \mu^B\,,\ U^B \sim \mathcal{U}_{[-1,1]}\} \\ &= max\{\mathbb{E}[< L, U >] \mid L \sim \mu\,,\ U\ bi-uniform\} \end{aligned}$$

where U is bi-uniform means $U^A \sim U^B \sim \mathcal{U}_{[-1,1]}$.
Value of Γ_n

$$V_n(\mu) = max\{\mathbb{E}[< L, (\sum_{k=1}^{n} U_k) >] \mid L \sim \mu,\ (U_k)_{k=1,..,n} \text{ bi-uniform sequence}\}$$

$(U_k)_{k=1,..,n}$ is a bi-uniform sequence if :

$$\forall k,\ law(U_k \mid U_1, ..U_{k-1}) \text{ is bi-uniform}$$

$$\Leftrightarrow \quad law(U_k^A \mid U_1, .., U_{k-1}) = law(U_k^B \mid U_1, .., U_{k-1}) = \mathcal{U}_{[-1,1]}$$

Convergence of the value V_n : The two assets case

$$\frac{1}{\rho\sqrt{n}}V_n(\mu) = max\mathbb{E}[< L, \frac{\sum_{k=1}^n U_k}{\rho\sqrt{n}} >]$$
$$= max\mathbb{E}[L^A \frac{\sum_{k=1}^n U_k^A}{\rho\sqrt{n}} + L^B \frac{\sum_{k=1}^n U_k^B}{\rho\sqrt{n}}]$$

Central limit result : The asymptotic distribution of $\frac{1}{\rho\sqrt{n}}(\sum_{k=1}^n U_k^A, \sum_{k=1}^n U_k^B)$ depends only on the covariance matrix :

$$\begin{pmatrix} 1 & c_k \\ c_k & 1 \end{pmatrix} \text{ with } c_k = \frac{1}{\rho}\mathbb{E}[U_k^A U_k^B \mid U_1, .., U_{k-1}] \in [-1, 1]$$

Intuitively, the asymptotic problem will be to find the optimal correlation process c_t.

Sketch of the proof

1) V is an upper bound : central limit result.

$$v_1(\mu^A) \leq M(\mu^A) = max\mathbb{E}[L^A S]$$

where the maximum is taken over the joint distributions of (L^A, S) such that :

- $L^A \sim \mu^A$
- $\mathbb{E}[S] = 0$, $\mathbb{E}[(S)^2] = 1$ and $\mathbb{E}[(S)^{p'}] \leq (4K + \rho)^{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

2) V is a lower bound : approximation of an optimal process for the problem V by discrete martingales whose V_1-variation is asymptotically optimal.

Sketch of the proof

P1 can control the a posteriori martingale :
Let $(L_k, \mathcal{F}_k)_{k=0,..,n+1} \in \mathcal{M}_n(\mu)$ defined on some probability space Ω.
Nature chooses $L = L_{n+1}$ and P1 observes the whole space Ω.
At stage k, P1 chooses i_k using only information in $\mathcal{F}_k$.
Conditional payoff at stage k :

$$\begin{aligned}\mathbb{E}[< L.\tilde{R}(i_k, j_k) > +\tilde{N}(i_k, j_k) \mid \mathcal{F}_{k-1}] \\ = \mathbb{E}[< L_k.\tilde{R}(i_k, j_k) > +\tilde{N}(i_k, j_k) \mid \mathcal{F}_{k-1}]\end{aligned}$$

P1 has not to care about the information he will reveal, and at each round he's facing the one step game $\Gamma_1([L_k \mid \mathcal{F}_{k-1}])$
Playing optimal strategies at each step, P1 guarantees :

$$\mathbb{E}[\sum_{k=1}^{n} V_1([L_k^{(n)} \mid \mathcal{F}_{k-1}])].$$

$$v_n(\mu^A) = max_{\mathcal{M}_n(\mu^A)}\mathbb{E}[\sum_{k=1}^{n} v_1([L_k^A \mid \mathcal{F}_{k-1}])]$$

$$= max_{\mathcal{M}_n(\mu^A)}\mathbb{E}[\sum_{k=1}^{n} \max_{[U_k^A|\mathcal{F}_{k-1}]\sim\mathcal{U}_{[-1,1]}} \mathbb{E}[L_k^A U_k^A \mid \mathcal{F}_{k-1}]]$$

$$= max_{\mathcal{M}_n(\mu^A)}\mathbb{E}[\sum_{k=1}^{n} \max_{[U_k^A|\mathcal{F}_{k-1}]\sim\mathcal{U}_{[-1,1]}} \mathbb{E}[L_{n+1}^A U_k^A \mid \mathcal{F}_{k-1}]]$$

$$= max\mathbb{E}[L_{n+1}^A(\sum_{k=1}^{n} U_k^A)]$$

where the last maximum is taken over joint distribution $(L_{n+1}^A, \sum_{k=1}^n U_k^A)$ such that :

- $L_{n+1}^A \sim \mu^A$
- $(U_1^A, .., U_n^A)$ i.i.d. sequence of $\mathcal{U}_{[-1,1]}$ random variables.