

The Continuous Time Nonzero-sum
Dynkin Game and Applications.

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Outlines

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1. Introduction

- $(\Omega, \mathcal{F}, (F_t)_{t \leq T}, P)$ is a filtered probability space with $(F_t)_{t \leq T}$ complete and right continuous ; T is the horizon of the problem.

- Two players a_1 and a_2 act on a system up to the time when one of them decides to stop controlling, at a stopping time τ_1 (resp. τ_2) for a_1 (resp. a_2).

- The reward for a_1 (resp. a_2) is given by

$$J_1(\tau_1, \tau_2) \stackrel{\Delta}{=} E^{\frac{1}{2}} \left[X_{\tau_1}^1 \mathbf{1}_{\{\tau_1 \leq \tau_2\}} + Y_{\tau_2}^1 \mathbf{1}_{\{\tau_2 < \tau_1\}} \right]^{\frac{3}{4}}$$

(resp.

$$J_2(\tau_1, \tau_2) \stackrel{\Delta}{=} E^{\frac{1}{2}} \left[X_{\tau_2}^2 \mathbf{1}_{\{\tau_2 < \tau_1\}} + Y_{\tau_1}^2 \mathbf{1}_{\{\tau_1 \leq \tau_2\}} \right]^{\frac{3}{4}}).$$

Definition: A pair (τ_1^*, τ_2^*) of F_t -stopping times is called a Nash equilibrium point for the NZSDG if it satisfies: $\forall \tau_1, \tau_2 \in \mathcal{T}_0$,

$$J_1(\tau_1, \tau_2^*) \leq J_1(\tau_1^*, \tau_2^*)$$

and

$$J_2(\tau_1^*, \tau_2) \leq J_2(\tau_1^*, \tau_2^*).$$

Particular case: $J_1 + J_2 = 0$ corresponds to the zero-sum Dunkin game and a NEP for the game is just a saddle point for the ZSDG. It satisfies: $\forall \tau_1, \tau_2$,

$$J_1(\tau_1^*, \tau_2) \leq J_1(\tau_1^*, \tau_2^*) \leq J_1(\tau_1, \tau_2^*).$$

2. Known Results

A. PDE approach (Bens.-Fried., 77)

Assume:

- $\zeta := (\zeta_t)_{t \leq T}$ is a solution of a standard differential equation whose generator is \mathcal{A}
- $X_t^i = \varphi^i(t, \zeta_t)$ and $Y^i = \psi^i(t, \zeta_t)$ where ψ^i and φ^i deterministic functions
- [H1]: $X^i \leq Y^i$
- [H2]: Y^i supermartingales.

Theorem: There exist two deterministic continuous bounded functions $u^1(t, x)$ and $u^2(t, x)$ solution of the following system:

$$\begin{aligned}
 & u^i(T, x) = \psi^i(T, x); \\
 & u^i \geq \varphi^i; \\
 & \text{if } u^j(t, x) = \varphi^j(t, x) \text{ for } j \neq i \\
 & \quad \text{and some } (t, x), \text{ then } u^i(t, x) = \psi^i(t, x); \\
 & \text{if } \Sigma^i = \{(t, x), u^j(t, x) > \varphi^j(t, x) \text{ for } j \neq i\}, \\
 & \quad \text{then } \mathcal{A}u^i(t, x) \geq 0 \text{ for } (t, x) \in \Sigma^i; \\
 & (u^i - \varphi^i) \cdot \mathcal{A}u^i(t, x) = 0 \text{ in } \Sigma^i
 \end{aligned} \tag{1}$$

and the following pair of stopping times,

$$\hat{\tau}_i = \inf\{s \geq 0, u^i(s, \zeta_s) = \varphi^i(s, \zeta_s)\} \wedge T; i = 1, 2$$

is a NEP for the NZSDG.

B. The probabilistic approach (Etourn., 86)

Theorem: The processes are general and satisfy [H1]-[H2]. Then the NZSDG has a NEP.

The proof uses the notion of Snell envelope of processes which is the following:

Let U be an RCLL adapted stochastic process. The Snell envelope of U , denoted by $R(U)$, is the smallest \mathcal{F}_t -supermartingale which dominates U , *i.e.*, if \bar{W} is another RCLL supermartingale such that $\bar{W}_t \geq U_t$ for all $0 \leq t \leq T$, then $\bar{W}_t \geq W_t$ for any $0 \leq t \leq T$.

It satisfies the following properties:

(i) For any \mathbb{F} -stopping time θ we have:

$$W_\theta = \operatorname{esssup}_{\tau \geq \theta} E[U_\tau | \mathcal{F}_\theta] \quad P - a.s. (W_T = U_T);$$

(ii) Assume that U has only positive jumps. Then the stopping time

$$\tau^* \triangleq \inf\{s \geq 0, W_s = U_s\} \wedge T$$

is optimal, *i.e.*,

$$E[W_0] = E[W_{\tau^*}] = E[U_{\tau^*}] = \sup_{\tau \in \mathcal{T}_0} E[U_\tau].$$

As a by-product we have $W_{\tau^*} = U_{\tau^*}$ and the process W is a martingale on the time interval $[0, \tau^*]$.

The main idea of Etourneau's proof is:

Let \mathcal{E}_1 (resp. \mathcal{E}_2) be the set of RCLL adapted processes V^1 (resp. V^2) such that $X^1 \leq V^1 \leq Y^1$ (resp. $X^2 \leq V^2 \leq Y^2$).

For $(i, j) = (1, 2)$ and for $V^j \in \mathcal{E}_j$

$$D_j = \inf\{s \geq 0, V_s^j = X_s^j\} \wedge T$$

and

$$f_i(V^j) = R(X^i 1_{[[0, D_j[[} + Y^i 1_{[[D_j, T]]}).$$

Then [H2] makes that:

(i) f_i is a map from \mathcal{E}_j to \mathcal{E}_i ,

(ii) $f_i(V^j) = R(V^j) 1_{[[0, D_j[[} + Y^i 1_{[[D_j, T]]}$

Therefore the mappings $f_1 \circ f_2$ and $f_2 \circ f_1$ have fixed points W^1 and W^2 which provide a NEP for the NZSDG whose expression is:

$$\tau^1 = \inf\{s \geq 0, W_s^1 = X_s^1\} \wedge T$$

and

$$\tau^2 = \inf\{s \geq 0, W_s^2 = X_s^2\} \wedge T.$$

3. The main result: without [H2].

Theorem: Assume:

- [H1] i.e. $X^1 \leq Y^1$ and $X^2 \leq Y^2$

- for any stopping time τ ,

$$P[\{X_\tau^1 < Y_\tau^1\} - \{X_\tau^2 < Y_\tau^2\}] = 0$$

(assumption which is satisfied if $X^2 < Y^2$).

- $X_T^1 = Y_T^1$ (technical and can be removed).

Then the NZSDG has a NEP (τ_1^*, τ_2^*) .

Sketch of the proof:

Let $\tau_1 = T$ and $\tau_2 = T$. For $n = 1, \dots$, assume τ_{2n-1} and τ_{2n} defined, we then define τ_{2n+1} and τ_{2n+2} as: Let

$$W_t^{2n+1} = \operatorname{esssup}_{\tau \geq t} E[X_\tau^1 \mathbf{1}_{\{\tau < \tau_{2n}\}} + Y_{\tau_{2n}}^1 \mathbf{1}_{\{\tau \geq \tau_{2n}\}} | F_t]$$

$$\tilde{\tau}_{2n+1} = \inf\{t \geq 0 : W_t^{2n+1} = X_t^1\} \wedge \tau_{2n}$$

and

$$\tau_{2n+1} \begin{cases} \tilde{\tau}_{2n+1}, & \text{if } \tilde{\tau}_{2n+1} < \tau_{2n}; \\ \tau_{2n-1}, & \text{if } \tilde{\tau}_{2n+1} = \tau_{2n}. \end{cases}$$

Next, let

$$W_t^{2n+2} = \text{esssup}_{\tau \geq t} E[X_\tau^2 \mathbf{1}_{\{\tau < \tau_{2n+1}\}} + Y_{\tau_{2n+1}}^2 \mathbf{1}_{\{\tau \geq \tau_{2n+1}\}} | F_t],$$

$$\tilde{\tau}_{2n+2} = \inf\{t \geq 0 : W_t^{2n+2} = X_t^2\} \wedge \tau_{2n+1}$$

and

$$\tau_{2n+2} = \begin{cases} \tilde{\tau}_{2n+2}, & \text{if } \tilde{\tau}_{2n+2} < \tau_{2n+1}; \\ \tau_{2n}, & \text{if } \tilde{\tau}_{2n+2} = \tau_{2n+1}. \end{cases}$$

The sequences $(\tau_{2n})_{n \geq 0}$ and $(\tau_{2n+1})_{n \geq 0}$ are decreasing and converge respectively to τ_1^* and τ_2^* respectively and (τ_1^*, τ_2^*) is a NEP for the NZSDG.

Step 1: for any stopping time τ ,

$$J_1(\tau, \tau_{2n}) \leq J_1(\tau_{2n+1}, \tau_{2n})$$

and

$$J_2(\tau_{2n+1}, \tau) \leq J_2(\tau_{2n+1}, \tau_{2n+2}).$$

By definition of W^{2n+1} ,

- $W_{\tau_{2n}}^{2n+1} = Y_{\tau_{2n}}^1$
- $W_t^{2n+1} \geq X_t^1$ for any $t \in [0, \tau_{2n}]$
- W^{2n+1} is a supermartingale over $[0, \tau_{2n}]$.

Then, for any τ ,

$$\begin{aligned}
 J_1(\tau, \tau_{2n}) &= E X_{\tau}^1 \mathbf{1}_{\{\tau \leq \tau_{2n}\}} + Y_{\tau_{2n}}^1 \mathbf{1}_{\{\tau_{2n} < \tau\}} \quad \frac{3}{4} \\
 &\leq E W_{\tau}^{2n+1} \mathbf{1}_{\{\tau \leq \tau_{2n}\}} + W_{\tau_{2n}}^{2n+1} \mathbf{1}_{\{\tau_{2n} < \tau\}} \\
 &= E\{W_{\tau_{2n} \wedge \tau}^{2n+1}\} \leq W_0^{2n+1}.
 \end{aligned}$$

But

$$\begin{aligned}
 J_1(\tau_{2n+1}, \tau_{2n}) &= \quad \frac{3}{4} \\
 E X_{\tau_{2n+1}}^1 \mathbf{1}_{\{\tau_{2n+1} \leq \tau_{2n}\}} + Y_{\tau_{2n}}^1 \mathbf{1}_{\{\tau_{2n} < \tau_{2n+1}\}} \quad \frac{3}{4} \\
 &= E X_{\tau_{2n+1}}^1 \mathbf{1}_{\{\tau_{2n+1} < \tau_{2n}\}} + Y_{\tau_{2n}}^1 \mathbf{1}_{\{\tau_{2n} \leq \tau_{2n+1}\}} \cdot
 \end{aligned}$$

Then

$$\begin{aligned}
 J_1(\tau_{2n+1}, \tau_{2n}) &= \quad \frac{3}{4} \\
 E X_{\tilde{\tau}_{2n+1}}^1 \mathbf{1}_{\{\tilde{\tau}_{2n+1} < \tau_{2n}\}} + W_{\tau_{2n}}^{2n+1} \mathbf{1}_{\{\tilde{\tau}_{2n+1} = \tau_{2n}\}} \\
 &= E\{W_{\tilde{\tau}_{2n+1}}^{2n+1}\} = W_0^{2n+1}.
 \end{aligned}$$

Therefore

$$J_2(\tau_{2n+1}, \tau) \leq J_2(\tau_{2n+1}, \tau_{2n+2}).$$

In the same way we have the other inequality.

Step 2: Take the limit to obtain the desired result.

4. Application in game options

Assume we have an American game contingent claim whose payoff is:

$$\Gamma(\tau, \sigma) = L\sigma \mathbf{1}_{[\sigma \leq \tau, \sigma < T]} + U\tau \mathbf{1}_{[\tau < \sigma]} + \xi \mathbf{1}_{[\tau = \sigma = T]}.$$

• $L \leq U$ and the difference $U - L$ is the compensation that a_1 pays to a_2 for the decision to terminate the contract before maturity date T .

In a complete market the value of the GCC is given by :

$$\begin{aligned} V_0 &= \sup_{\sigma \geq 0} \inf_{\tau \geq 0} E^*[\Gamma(\tau, \sigma)] \\ &= \inf_{\tau \geq 0} \sup_{\sigma \geq 0} E^*[\Gamma(\tau, \sigma)]. \end{aligned}$$

In incomplete markets another point of view is related to utility maximization of the agents (Kuhn, 03).

Let $\varphi_1, \varphi_2 : \mathbb{R} \rightarrow \mathbb{R}$ be two utility functions of the seller, respectively, the buyer of the GCC. The seller a_1 (resp. the buyer a_2) chooses a stopping time τ (resp. σ) in order to maximize

$$J_1(\tau, \sigma) := E[\varphi_1(-\Gamma(\tau, \sigma))]$$

(resp.

$$J_2(\tau, \sigma) := E[\varphi_2(\Gamma(\tau, \sigma))]).$$

Therefore if the NZSDG associated with J_1 and J_2 has a NEP point (σ^*, τ^*) , i.e.,

$$J_1(\tau^*, \sigma^*) \geq J_1(\tau, \sigma^*) \text{ and } J_2(\tau^*, \sigma^*) \geq J_2(\tau^*, \sigma)$$

then $-\varphi_1^{-1}(J_1(\tau^*, \sigma^*))$ (resp. $\varphi_2^{-1}(J_2(\tau^*, \sigma^*))$) is a seller (resp. buyer) price of the GCC.

Theorem: Assume that:

(i) The utility functions φ_1 and φ_2 are non-decreasing;

(ii) L , U are continuous and $L_t \leq U_t$ and $L_T \leq \xi \leq U_T$, P-a.s.;

Then the nonzero-sum Dynkin game associated with the GCC has a Nash equilibrium point (τ^*, σ^*) .

Thanks a lot for your attention.