Differential games with lack of information

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Joint works with C. Rainer (Brest)

Dynamic games, Differential games III
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Outline

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Dynamics

We investigate a stochastic differential game defined by

$$\begin{cases} dX_t = f(X_t, u_t, v_t)dt + \sigma(X_t, u_t, v_t)dB_t, \ t \in [t_0, T], \\ X_{t_0} = x_0, \end{cases}$$

where

- B is a d-dimensional standard Brownian motion
- $f : \mathbb{R}^N \times U \times V \to \mathbb{R}^N$ and $\sigma : \mathbb{R}^n \times U \times V \to \mathbb{R}^{N \times d}$ are Lipschitz continuous and bounded,
- the processes u (controlled by Player I) and v (controlled by Player II) take their values in some compact sets U and V.

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- the processes u (controlled by Player I) and v (controlled by Player II) take their values in some compact sets U and V.

The terminal payoff

Let $g : \mathbb{R}^N \to \mathbb{R}$ a terminal payoff,

- Player I tries to minimise the terminal payoff $\mathbf{E}[g(X_T)]$
- Player II tries to maximise the terminal payoff $\mathbf{E}[g(X_T)]$

- The players observe the position of the state.

Outline

1. Solving classical differential games
   - Description of the game
   - Formalisation
   - Existence of the value
2. Differential games with lack of information
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   - Existence and characterization of the value
3. A new formulation for dual solutions
   - A strange HJ equation
   - Illustration by a simple game
4. Appendix : without Isaacs' condition

Admissible controls

For $t_0 \in [0, T[$, we set

$$\mathcal{F}_{t_0,s} = \sigma\{B_r - B_{t_0}, r \in [t, s]\} \vee \mathcal{P},$$

where $\mathcal{P}$ is the set of all null-sets of P.

- An admissible control for player I on $[t_0, T]$ is a process $u : [t_0, T] \to U$ progressively measurable with respect to $(\mathcal{F}_{t_0,s}, s \geq t_0)$.

$$\mathcal{U}(t_0) = \{u \text{ admissible control on } [t_0, T]\ \}\ .$$

- the set of admissible controls of Player II is defined symmetrically and denoted by $\mathcal{V}(t_0)$.

Controls and dynamics

For $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ and an initial data $x_0 \in \mathbb{R}^N$ at time t_0, we denote by

$$t \to X_t^{t_0, x_0, u, v}$$

the solution to

$$\left\{ \begin{array}{l} dX_t = f(t, X_t, u_t, v_t)ds + \sigma(t, X_t, u_t, v_t)dB_t, \ t \in [t_0, T], \\ X_{t_0} = x_0, \end{array} \right.$$

Pure strategies

- A pure strategy for Player I is a Borel measurable map $\alpha : [t_0, T] \times \mathcal{C}^0([t_0, T], \mathbb{R}^N) \to U$ such that there is $\tau > 0$ with

$$f_1 = f_2 \text{ on } [t_0, t] \quad \Rightarrow \quad \alpha(s, f_1) = \alpha(s, f_2) \text{ for } s \in [t_0, t + \tau]$$

  The set of pure strategies for Player I is denoted by $\mathcal{A}(t_0)$.
- The set of pure strategies for Player II is defined symmetrically and denoted by $\mathcal{B}(t_0)$.

Playing pure strategies together

Lemma

For all $(t_0, x_0) \in [0, T] \times \mathbb{R}^N$, *for all* $(\alpha, \beta) \in \mathcal{A}(t_0) \times \mathcal{B}(t_0)$, *there exists a unique couple of controls* $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ *that satisfies*

$$(*) \qquad (u, v) \equiv (\alpha(\cdot, X_\cdot^{t_0,x,u,v}), \beta(\cdot, X_\cdot^{t_0,x,u,v})) \text{ on } [t_0, T].$$

Notation : $X_t^{t_0,x_0,\alpha,\beta} := X_t^{t_0,x_0,u,v}$ where (u, v) is given by $(*)$.

Upper and lower value functions

The upper value function is

$$V^+(t_0, x_0) = \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{\beta \in \mathcal{B}(t_0)} \mathbf{E}\left[g(X_T^{t_0,x_0,\alpha,\beta})\right]$$

while the lower value function is

$$V^-(t_0, x_0) = \sup_{\beta \in \mathcal{B}(t_0)} \inf_{\alpha \in \mathcal{A}(t_0)} \mathbf{E}\left[g(X_T^{t_0,x_0,\alpha,\beta})\right]$$

Outline

Isaacs' condition

We assume that Isaacs'condition holds : for all $(t,x) \in [0,T] \times \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, and all $A \in \mathcal{S}_n$:

$$H(x,\xi,A) :=$$
$$\inf_u \sup_v \{< f(x,u,v), \xi > + \frac{1}{2} Tr(A\sigma(x,u,v)\sigma^*(x,u,v))\}$$
$$= \sup_v \inf_u \{< f(x,u,v), \xi > + \frac{1}{2} Tr(A\sigma(x,u,v)\sigma^*(x,u,v))\}$$

Existence of a value

Theorem (Fleming-Souganidis, 1989)

Under Isaacs' condition, the game has a value :

$$V^+(t,x) = V^-(t,x) \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^N .$$

Sketch of proof (1)

Lemma

The value functions V^+ and V^- are Hölder continuous in $[0, T] \times \mathbb{R}^N$.

Lemma (Dynamic programming)

For $(t_0, x_0) \in [0, T] \times \mathbb{R}^N$ and $h > 0$,

$$V^+(t_0, x_0) \geq \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{\beta \in \mathcal{B}(t_0)} \mathbf{E}\left[V^+\left(t_0 + h, X_{t_0+h}^{t_0, x_0, \alpha, \beta}\right)\right]$$

and

$$V^-(t_0, x_0) \leq \sup_{\beta \in \mathcal{B}(t_0)} \inf_{\alpha \in \mathcal{A}(t_0)} \mathbf{E}\left[V^-\left(t_0 + h, X_{t_0+h}^{t_0, x_0, \alpha, \beta}\right)\right]$$

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Sketch of proof (2)

Let $\varphi = \varphi(t,x)$ be a smooth test function such that $V^- - \varphi$ has a minimum at (t_0, x_0). Then

$$V^-(t,x) - V^-(t_0,x_0) \geq \varphi(t,x) - \varphi(t_0,x_0) \qquad \forall (t,x)\ .$$

From dynamic programming :

$$0 \geq \sup_{\beta\in\mathcal{B}(t_0)} \inf_{\alpha\in\mathcal{A}(t_0)} \mathbf{E}\left[V^-\left(t_0+h, X_{t_0+h}^{t_0,x_0,\alpha,\beta}\right)\right] - V^-(t_0,x_0)$$

$$\geq \sup_{\beta\in\mathcal{B}(t_0)} \inf_{\alpha\in\mathcal{A}(t_0)} \mathbf{E}\left[\varphi\left(t_0+h, X_{t_0+h}^{t_0,x_0,\alpha,\beta}\right) - \varphi(t_0,x_0)\right]$$

$$\approx \sup_{\beta\in\mathcal{B}(t_0)} \inf_{\alpha\in\mathcal{A}(t_0)} \left\{h\varphi_t + \int_{t_0}^{t_0+h} D\varphi.f(\alpha,\beta) + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(\alpha,\beta)D^2\varphi)ds\right\}$$

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Sketch of proof (3)

So

$$0 \geq \varphi_t + \sup_{v\in V} \inf_{u\in U} \left\{ < D\varphi, f(x_0,u,v) > + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(x,u,v)D\varphi) \right\}$$
$$= \varphi_t + H(x_0, D\varphi, D^2\varphi) \qquad \text{at } (t_0,x_0) \ .$$

Definition

A continuous map $w : [0,T]\times\mathbb{R}^N \to \mathbb{R}$ is a viscosity supersolution of the Hamilton-Jacobi equation

$$(HJI) \qquad w_t + H(x, Dw, D^2w) = 0$$

if, for any smooth test function $\varphi = \varphi(t,x)$ such that $w-\varphi$ has a minimum at (t_0,x_0),

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Finally w is a viscosity solution if w is a super- and a sub-solution.

Proposition

V^+ is a subsolution and V^- is a supersolution of (HJI) and

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Sketch of proof (5)

Theorem (Comparison principle)

If w_1 is a subsolution and w_2 is a supersolution and if

$$w_1(T,x) \leq w_2(T,x) \qquad \forall x \in \mathbb{R}^N ,$$

then

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Proof of the existence of a value : By comparison principle,

$$V^+ \leq V^- .$$

Since $V^- \leq V^+$ always holds, $V^+ = V^-$. Note that $V^+ = V^-$ is the unique solution of the HJI equation.

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The terminal payoffs

Let

- $g_{ij} : \mathbb{R}^N \to \mathbb{R}$ a family of terminal payoffs, $i = 1, \ldots, I$, $j = 1, \ldots, J$
- $p \in \Delta(I)$ be a probability on $\{1, \ldots, I\}$.
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Organization of the game

The game is played in two steps :

- At initial time t_0 the pair (i, j) is chosen at random according to probability $p \otimes q$.
  Index i is communicated to Player I only, while
  index j is communicated to Player II only.
- Then
  - Player I tries to minimise the terminal payoff $\mathbf{E}[g_{ij}(X_T)]$
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Key assumptions on the game

The Players observe the state (X_t).

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Random strategies

- A random strategy for Player I is $R-$uple

$$\overline{\alpha} = (\alpha^1, \dots \alpha^R; r^1, \dots, r^R) \ ,$$

with $R \in \mathbb{N}^*$, $\alpha^1, \dots, \alpha^R \in \mathcal{A}(t)$, $(r^1, \dots, r^R) \in \Delta(R)$.

Interpretation : Player I choses at random according to probability $r = (r^1, \dots, r^R)$ a strategy $\alpha^1, \dots, \alpha^R$.

- Notations : The set of random strategies for Player I (resp. Player II) is denoted by $\mathcal{A}_r(t)$ (resp. $\mathcal{B}_r(t)$).

Admissible strategies

Remark : Since Player I knows i, he can chose a strategy which depends on i.

- So an admissible strategy for Player I is an element $\hat{\alpha} = (\overline{\alpha}_1, \dots, \overline{\alpha}_I) \in (\mathcal{A}_r(t))^I$.
- Symmetrically an admissible strategy for Player II is an element $\hat{\beta} = (\overline{\beta}_1, \dots, \overline{\beta}_J) \in (\mathcal{B}_r(t))^J$

Payoff associated with two admissible strategies

- For $(\overline{\alpha}, \overline{\beta}) \in \mathcal{A}_r(t) \times \mathcal{B}_r(t)$, with

$$\overline{\alpha} = ((\alpha^1, \dots \alpha^R; r^1, \dots, r^R) \text{ and } \overline{\beta} = ((\beta^1, \dots \beta^S; s^1, \dots, s^S)$$

we set

$$J_{ij}(t_0, x_0, \overline{\alpha}, \overline{\beta}) = \sum_{k,l} r^k s^l \mathbf{E}\left[g_{ij}(X_T^{t_0,x_0,\alpha^k,\beta^l})\right]$$

- For $p \in \Delta(I)$, $q \in \Delta(J)$, $\hat{\alpha} \in (\mathcal{A}_r(t))^I$ and $\hat{\beta} \in (\mathcal{B}_r(t))^J$ with

$$\hat{\alpha} = (\overline{\alpha}_1, \dots, \overline{\alpha}_I) \text{ and } \hat{\beta} = (\overline{\beta}_1, \dots, \overline{\beta}_J)$$

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$$J(t_0, x_0, \hat{\alpha}, \hat{\beta}, p, q) = \sum_{i,j} p_i q_j J_{ij}(t_0, x_0, \overline{\alpha}_i, \overline{\beta}_j)$$

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Upper- and lower value functions

The upper value function is

$$V^+(t_0, x_0, p, q) = \inf_{\hat{\alpha}\in(\mathcal{A}_r(t_0))^I} \sup_{\hat{\beta}\in(\mathcal{B}_r(t_0))^J} J(t_0, x_0, \hat{\alpha}, \hat{\beta}, p, q)$$

while the lower value function is

$$V^-(t_0, x_0, p, q) = \sup_{\hat{\beta}\in(\mathcal{B}_r(t_0))^J} \inf_{\hat{\alpha}\in(\mathcal{A}_r(t_0))^I} J(t_0, x_0, \hat{\alpha}, \hat{\beta}, p, q)$$

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We assume that Isaacs'condition holds : for all $(t,x) \in [0,T] \times \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, and all $A \in \mathcal{S}_n$:

$$H(x,\xi,A) :=$$
$$\inf_u \sup_v \{< f(x,u,v), \xi > + \frac{1}{2} Tr(A\sigma(x,u,v)\sigma^*(x,u,v))\}$$
$$= \sup_v \inf_u \{< f(x,u,v), \xi > + \frac{1}{2} Tr(A\sigma(x,u,v)\sigma^*(x,u,v))\}$$

Existence of a value

Theorem (C.-Rainer, To appear)

Under Isaacs' condition, the game has a value :
$\forall (t,x,p,q) \in [0,T] \times \mathbb{R}^N \times \Delta(I) \times \Delta(J)$

$$V^+(t,x,p,q) = V^-(t,x,p,q) \ .$$

Main difficulties for the proof

- Existence of a value for classical differential games is based on
  - a dynamic programming implying that both value functions satisfy the same HJI equation
  - uniqueness of this equation
- In our game, Players learn a part of their missing information along the time :

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Regularity and convexity of the value functions

Lemma

V^+ and V^- are bounded, Lipschitz continuous with respect to x and Hölder continuous with respect to t.

Proposition

For all $(t,x) \in [0,T] \times \mathbb{R}^n$, the maps $(p,q) \to V^+(t,x,p,q)$ and $(p,q) \to V^-(t,x,p,q)$ are convex in p and concave in q.

Proof : Splitting method.

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Proof : Splitting method.

Fenchel conjugates

As in [De Meyer, 1996] we introduce the Fenchel conjugates of V^+ and V^- :

$$(V^{\pm})^*(t,x,\hat{p},q) = \sup_{p\in\Delta(I)} \left(p.\hat{p} - V^{\pm}(t,x,p,q)\right)$$

and

$$(V^{\pm})^\sharp(t,x,p,\hat{q}) = \inf_{q\in\Delta(J)} \left(q.\hat{q} - V^{\pm}(t,x,p,q)\right)$$

Reformulation for the conjugates

Lemma

For all $(t, x, \hat{p}, q) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^I \times \Delta(J)$, we have

$$V^{-*}(t, x, \hat{p}, q) = \inf_{\hat{\beta} \in (\mathcal{B}_r(t))^J} \sup_{\alpha \in \mathcal{A}(t)} \max_{i \in \{1,\dots,I\}} \left\{ \hat{p}_i - \sum_j q_j J_{ij}(t, x, \alpha, \overline{\beta}_j) \right\}.$$

Subdynamic programming for V^{-*}

Proposition

For all $0 \leq t_0 \leq t_1 \leq T, x_0 \in \mathbb{R}^N, \hat{p} \in \mathbb{R}^I, q \in \Delta(J)$,

$$V^{-*}(t_0, x_0, \hat{p}, q) \leq \inf_{\beta \in \mathcal{B}(t_0)} \sup_{\alpha \in \mathcal{A}(t_0)} \mathbf{E}[V^{-*}(t_1, X_{t_1}^{t_0,x_0,\alpha,\beta}, \hat{p}, q)]$$

Idea of proof : If Player II plays a pure strategy β independant of j between t_0 and t_1,

- his payoff is larger,
- but he reveals nothing on j

So the game can be restarted at t_1 without loss of information.

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Equation satisfied by V^{-*}

Corollary

For any $(\hat{p}, q) \in \mathbb{R}^I \times \Delta(J)$, $(t, x) \to V^{-*}(t, x, \hat{p}, q)$ is a subsolution in viscosity sense of

$$(HJI*) \qquad w_t - H(x, -Dw, -D^2 w) = 0,$$

Definition (Supersolution in the dual sense)

We say that $w = w(t, x, p, q)$ is a viscosity supersolution of

$$(HJI) \qquad w_t + H(x, Dw, D^2 w) = 0 \qquad \text{in } (0, T) \times \mathbb{R}^N$$

in the dual sense if for any $(\hat{p}, q) \in \mathbb{R}^I \times \Delta(J)$, $(t, x) \to w^*(t, x, \hat{p}, q)$ is a subsolution of (HJI^*).

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Superdynamic programming for $V^{+\sharp}$

Proposition

For all $0 \le t_0 \le t_1 \le T, x_0 \in \mathbb{R}^N, p \in \Delta(I), \hat{q} \in \mathbb{R}^J$,

$$V^{+\sharp}(t_0, x_0, p, \hat{q}) \ge \sup_{\alpha \in \mathcal{A}(t_0)} \inf_{\beta \in \mathcal{B}(t_0)} E[V^{+\sharp}(t_1, X_{t_1}^{t_0, x_0, \alpha, \beta}, p, \hat{q})].$$

Hence $V^{+\sharp}$ is a supersolution of $(HJI*)$.

Definition (Subsolution in the dual sense)

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in the dual sense if for any $(p, \hat{q}) \in \Delta(I) \times \mathbb{R}^J$, $(t, x) \to w^\sharp(t, x, p, \hat{q})$ is a subsolution of (HJI^*).

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Comparison principle

Theorem

Let $w_1, w_2 : [0,T] \times \mathbb{R}^N \times \Delta(I) \times \Delta(J) \to \mathbb{R}$ be bounded, Hölder continuous, and uniformly Lipschitz continuous with respect to p and q. If

- w_1 is a subsolution of (HJI)in the dual sense,
- w_2 be a supersolution of (HJI) in the dual sense,
- $w_1(T,x,p,q) \leq w_2(T,x,p,q) \qquad \forall (x,p,q)$,

then

$$w_1(t,x,p,q) \leq w_2(t,x,p,q) \qquad \forall (t,x,p,q) \ .$$

Proof of the existence of the value

- We have $V^- \leq V^+$ by construction.
- We have seen that
  (i) V^- is a supersolution in the dual sense of (HJI)
  (ii) V^+ is a subsolution in the dual sense of (HJI)
  (iii) $V^-(T,x,p,q) = V^+(T,x,p,q) = \sum_{i,j} p_i q_j g_{ij}(x)$
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Extensions and miscellaenous

- Extensions The above results have been extended to
  - stochastic differential games with running payoff, (C. and Rainer)
  - infinite horizon problem (As Soulaimani)
- Representation formulas for deterministic differential games (C., Souquière)
- Approximation of the value function, $\epsilon-$optimal strategies for deterministic differential games (C., Souquière)
- Open problem : Pursuit-evasion games with lack of information.

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Outline

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Up to now, the value function $\mathbf{V} = V^+ = V^-$ is characterized by the fact that V^* and $V^\sharp$ are sub- and super-solutions of some dual HJI equation.

What about a direct characterization of $\mathbf{V}$?

Reformulation of the HJ equation

Theorem (C., 2008)

A map w is a dual solution of

$$w_t + H(x, Dw, D^2 w, p, q) = 0$$

if and only if w is a viscosity solution of the strange HJ equation

$$\max\left\{\min\left\{w_t + H(x, Dw, D^2 w, p, q)\ ;\ \lambda_{\min}\left(\frac{\partial^2 w}{\partial p^2}\right)\right\}\ ;\ \lambda_{\max}\left(\frac{\partial^2 w}{\partial q^2}\right)\right\} = 0$$

where

- $\lambda_{\max}(A)$ is the maximal eigenvalue of a matrix $A \in \mathcal{S}_k$
- $\lambda_{\min}(A)$ is the minimal eigenvalue of A

In particular,

Corollary

The value function $\mathbf{V}$ *of the game with lack of information is the unique viscosity solution of*

$$\max\left\{\min\left\{w_t + H(x, Dw, D^2w, p, q)\ ;\ \lambda_{\min}\left(\frac{\partial^2 w}{\partial p^2}\right)\right\}\ ;\ \lambda_{\max}\left(\frac{\partial^2 w}{\partial q^2}\right)\right\} = 0$$

Remarks on the strange equation :

1) From the min-max Theorem we have

$$\max\{\min\{\dots\ ;\dots\}\ ;\ \dots\} = \min\{\max\{\dots\ ;\dots\}\ ;\ \dots\}$$

2) Heuristically this equation says that

- the map $\mathbf{V} = \mathbf{V}(t, x, p, q)$ is convex in p and concave in q,
- at points where $\mathbf{V} = \mathbf{V}(t, x, p, q)$ is strictly convex in p and strictly concave in q, $\mathbf{V}$ satisfies the Hamilton-Jacobi equation

$$w_t + H(x, Dw, D^2 w) = 0$$

Example 1 : convex case

Let $g : \Delta(I) \to \mathbb{R}$ be Lipschitz continuous. Then the unique solution to

$$\max\left\{ w - g \; ; \; -\lambda_{\min}\left(\frac{\partial^2 w}{\partial p}\right)\right\} = 0$$

is just the convex hull of g.

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Example 2 : Mertens-Zamir Φ operator

Let $g : \Delta(I) \times \Delta(J) \to \mathbb{R}$ be Lipschitz continuous. Then the unique solution of

$$(*) \qquad \min\left\{\max\left\{w - g \,;\, -\lambda_{\min}\left(\frac{\partial^2 w}{\partial p}\right)\right\} \,;\, -\lambda_{\max}\left(\frac{\partial^2 w}{\partial q}\right)\right\} = 0$$

is $\Phi(g)$, i.e., the unique solution u to

$$u = \mathrm{Vex}_p(\max\{u \,;\, g\}) = \mathrm{Cav}_q(\min\{u \,;\, g\})$$

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Accordingly, the strange equation

$$\max\left\{\min\left\{w_t + H(x, Dw, D^2w, p, q)\ ;\ \lambda_{\min}\left(\frac{\partial^2 w}{\partial p^2}\right)\right\}\ ;\ \lambda_{\max}\left(\frac{\partial^2 w}{\partial q^2}\right)\right\} = 0$$

is between a standard HJ equation and a characterization of convexity.

Questions

1. Is there an interpretation of the strange equation in terms of dynamic programming ?
2. What is the set where

$$w_t + H(x, Dw, D^2w, p, q) = 0$$

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Outline

Definition of the simple game

In this game,

- no dynamics
- $J = 1$.

The players optimize one of the integral payoffs

$$\int_{t_0}^{T} \ell_i(s, u(s), v(s))ds \qquad (i \in \{1, \dots, I\}).$$

Rules of the game

- At time t_0, i is chosen by nature in $\{1, \ldots, I\}$ according to probability p,
- the choice of i is communicated to Player 1 only,
- Player 1 minimizes the integral payoff

$$\int_{t_0}^{T} \ell_i(s, u(s), v(s))ds.$$

- Player 2 maximizes it.

This is a version of Aumann-Maschler game in continuous time, finite horizon.

Isaacs'condition

Isaacs'condition takes the form :

$$H(t,p) = \inf_{u\in U}\sup_{v\in V}\sum_{i=1}^{I} p_i\ell_i(t,u,v) = \sup_{v\in V}\inf_{u\in U}\sum_{i=1}^{I} p_i\ell_i(t,u,v)$$

for all $(t,p)\in[0,T]\times\Delta(I)$.

Existence of a value

Theorem (C.-Rainer, 2008)

Under Isaacs' condition, the game has a value

$$\mathbf{V}(t_0,p) = \inf_{(\alpha_i)\in(\mathcal{A}_r(t_0))^I} \sup_{\beta\in\mathcal{B}_r(t_0)} \sum_{i=1}^{I} p_i \mathbf{E}_{\alpha_i\beta}\left[\int_{t_0}^{T} \ell_i(s,\alpha_i(s),\beta(s))ds\right]$$

$$= \sup_{\beta\in\mathcal{B}_r(t_0)} \inf_{(\alpha_i)\in(\mathcal{A}_r(t_0))^I} \sum_{i=1}^{I} p_i \mathbf{E}_{\alpha_i\beta}\left[\int_{t_0}^{T} \ell_i(s,\alpha_i(s),\beta(s))ds\right]$$

Furthermore **V** is the unique viscosity solution of :

$$\min\left\{w_t + H(t,p)\,;\ \lambda_{\min}\left(\frac{\partial^2 w}{\partial p^2}\right)\right\} = 0 \qquad \text{in } [0,T]\times\Delta(I)\,.$$

A set of admissible martingales

Let $\mathcal{P}(t_0, p_0)$ be the set of càdlàg martingale processes

$$\mathbf{p} : [t_0^-, T] \to \Delta(I)$$

such that

$$\mathbf{p}(t_0^-) = p_0 \qquad \text{and} \qquad \mathbf{p}(T) \in \{e_1, \ldots, e_I\} \ ,$$

where $\{e_1, \ldots, e_I\}$ is the canonical basis of $\mathbb{R}^I$.

Representation of the solution

Theorem

$$\mathbf{V}(t_0, p_0) = \inf_{\mathbf{p} \in \mathcal{P}(t_0, p_0)} \mathbf{E}\left[\int_{t_0}^{T} H(s, \mathbf{p}(s))ds\right] \qquad \forall (t_0, p_0) \in [0, T] \times \Delta(I) ,$$

Remark : A similar result in discrete time appears in [De Meyer, 2008].

Sketch of the proof

Let

$$W(t_0, p_0) = \inf_{\mathbf{p}\in\mathcal{P}(t_0,p_0)} \mathbf{E}\left[\int_{t_0}^{T} H(s, \mathbf{p}(s))ds\right] \qquad \forall (t_0, p_0) \in [0, T] \times \Delta(I)$$

Lemma

W is convex with respect to p_0 and Lipschitz continuous in all variables.

Lemma

For any stopping time $\theta \in [t_0, T]$,

$$W(t_0, p_0) = \inf_{\mathbf{p}\in\mathcal{P}(t_0,p_0)} \mathbf{E}\left[\int_{t_0}^{\theta} H(s, \mathbf{p}(s))ds + W(\theta, \mathbf{p}(\theta))\right] .$$

Sketch of the proof (2)

Lemma

W is a solution of

$$\min\left\{w_t + H(t,p)\,;\ \lambda_{\min}\left(\frac{\partial^2 w}{\partial p^2}\right)\right\} = 0 \qquad \text{in } [0,T]\times\Delta(I)\,.$$

Heuristic idea : At a point (t_0, x_0) at which "$\lambda_{\min}\left(\frac{\partial^2 w}{\partial p^2}\right) > 0$", the martingale process cannot "go too far from x_0" and the classical dynamic programming holds.

Consequences

- Characterization of the optimal strategy of the informed player.
- Characterization of the optimal martingale process.

Some open problems for differential games with lack of information

- One can prove the existence of a value when **i** and **j** belong to some continuous probability spaces.
    - What becomes the HJ equation in this setting ?

- More complex information structure
    - What happens if the players have a private information which is revealed along the time ?

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Outline

Random strategies

Fix an initial time t_0 and a delay $\tau >$.

- A pure strategy with delay τ for Player I is a Borel measurable map $\alpha : [t_0, T] \times \mathcal{C}^0([t_0, T], \mathbb{R}^N) \to U$ such that

$$f_1 = f_2 \text{ on } [t_0, t] \quad \Rightarrow \quad \alpha(s, f_1) = \alpha(s, f_2) \text{ for } s \in [t_0, t+\tau]$$

  Notation : $\mathcal{A}_\tau(t_0)$.
- A random strategy with delay with delay τ for Player I is $R-$uple

$$\overline{\alpha} = (\alpha^1, \dots \alpha^R; r^1, \dots, r^R) \;,$$

  with $R \in \mathbb{N}^*$, $\alpha^1, \dots, \alpha^R \in \mathcal{A}_\tau(t_0)$, $(r^1, \dots, r^R) \in \Delta(R)$.
- Notations : The set of random strategies with delay τ for Player I (resp. Player II) is denoted by $\mathcal{A}_{\tau r}(t_0)$ (resp. $\mathcal{B}_{\tau r}(t_0)$).

Upper and lower value functions

The upper value function is

$$V^+(t_0, x_0) = \lim_{\tau \to 0} \inf_{\alpha \in \mathcal{A}_{\tau r}(t_0)} \sup_{\beta \in \mathcal{B}_{\tau r}(t_0)} \mathbf{E}\left[g(X_T^{t_0,x_0,\alpha,\beta})\right]$$

while the lower value function is

$$V^-(t_0, x_0) = \lim_{\tau \to 0} \sup_{\beta \in \mathcal{B}_{\tau r}(t_0)} \inf_{\alpha \in \mathcal{A}_{\tau r}(t_0)} \mathbf{E}\left[g(X_T^{t_0,x_0,\alpha,\beta})\right]$$

Existence of a value

Theorem

Under Isaacs' condition, the game has a value :

$$V^+(t,x) = V^-(t,x) \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^N .$$

Krasovskii-Subbotin, 1988, for the determinist case ($\sigma \equiv 0$).

Characterization of the value

Let

$$H(x,p,A)$$
$$= \inf_{\mu\in\Delta(U)} \sup_{\nu\in\Delta(V)} \int_{U\times V} < p, f(x,u,v) > +\frac{1}{2}\mathrm{Tr}(A\sigma\sigma^*(x,u,v))\, d\mu(u)d\nu(v)$$
$$= \sup_{\nu\in\Delta(V)} \inf_{\mu\in\Delta(U)} \int_{U\times V} < p, f(x,u,v) > +\frac{1}{2}\mathrm{Tr}(A\sigma\sigma^*(x,u,v))\, d\mu(u)d\nu(v)$$

Theorem

$V^+ = V^-$ is the unique solution of

$$\begin{cases} w_t + H(x, Dw, D^2w) = 0 & \text{in } (0,T)\times\mathbb{R}^N \\ w(T,x) = g(x) & \forall x\in\mathbb{R}^N \end{cases}$$