Roscoff, Novembre 2008

2-Person Zero-Sum Stochastic Differential Games

based on common work with

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Objective of the talk

Generalization of the results of the pioneering work of Fleming and Souganidis on zero-sum two-player SDGs:

- cost functionals defined through controlled BSDEs
- the admissible control processes can depend on events occurring before the beginning of the game.

This latter extension has the consequence that the cost functionals become random. However, by making use of Girsanov transformation we prove that the upper and the lower value functions of the game remain deterministic. This approach combined with the BSDE method allows to get in a direct way:

upper and lower value functions are deterministic
$\longrightarrow$ Dynamic Programming Principle
$\longrightarrow$ Hamilton-Jacobi-Bellman-Isaacs equations.

At the end of the talk: some remarks on extensions of the above SDGs: SDGs defined through reflected BSDEs.

Preliminaries. Framework

$(\Omega, \mathcal{F}, P)$ canonical Wiener space: for a given finite time horizon $T > 0$,

- $\Omega = C_0([0,T];\mathbb{R}^d)$ (endowed with the supremum norm);
- $B_t(\omega) = \omega(t), t \in [0,T], \omega \in \Omega$ - the coordinate process;
- P - the Wiener measure on $(\Omega, \mathcal{B}(\Omega))$: unique probability measure w.r.t. B is a standard BM;
- $\mathcal{F} = \mathcal{B}(\Omega) \vee \mathcal{N}_P$;
- $\mathbb{F} = (\mathcal{F})_t)_{t\in[0,T]}$ with $\mathcal{F}_t = \mathcal{F}_t^B = \sigma\{B_s, s \le t\} \vee \mathcal{N}_P$.

$(\Omega, \mathcal{F}, \mathbb{F}, P; B)$ - the complete, filtered probability space on which we will work.

Dynamics of the game:

Initial data: $t \in [0,T]$, $\zeta \in L^2(\Omega, \mathcal{F}_t, P; R^d)$;
associated doubly controlled stochastic system:

$$\begin{array}{rcl} dX_s^{t,\zeta;u,v} & = & b(s, X_s^{t,\zeta;u,v}, u_s, v_s)ds + \sigma(s, X_s^{t,\zeta;u,v}, u_s, v_s)dB_s, \\ X_t^{t,\zeta;u,v} & = & \zeta, \qquad\qquad\qquad\qquad\qquad\qquad s \in [t,T], \end{array} \tag{1}$$

Player I: $u \in \mathcal{U} =: L^0_{\mathbb{F}}(0,T;U)$;
Player II: $v \in \mathcal{V} =: L^0_{\mathbb{F}}(0,T;V)$; U, V - compact metric spaces;
the mappings

$$\begin{array}{l} b : [0,T] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^n, \\ \sigma : [0,T] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^{n\times d}, \end{array}$$

are supposed to be continuous over $\mathbb{R}^n \times U \times V$ (for simplicity); Lipschitz in x, uniformly w.r.t (t,u,v); $b(t,0,u,v), \sigma(t,0,u,v)$ are bounded.
Existence and uniqueness of the solution $X^{t,\zeta,u,v} \in \mathcal{S}^2_{\mathbb{F}}(t,T;R^n)$.

Definition of the cost functionals

The cost functional is defined with the help of a backward SDE (BSDE):

Associated with $(t,\zeta) \in [0,T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n), u \in \mathcal{U}$ and $v \in \mathcal{V}$, we consider the BSDE:

$$\begin{array}{rcl} dY_s^{t,\zeta;u,v} & = & -f(s, X_s^{t,\zeta;u,v}, Y_s^{t,\zeta;u,v}, Z_s^{t,\zeta;u,v}, u_s, v_s)ds + Z_s^{t,x\zeta;u,v} dB_s, \\ Y_T^{t,\zeta;u,v} & = & \Phi(X_T^{t,\zeta;u,v}), \qquad\qquad s \in [t,T], \end{array} \tag{2}$$

where

◇ Final cost: $\Phi : \mathbb{R}^n \to \mathbb{R}$ Lipschitz

◇ Running cost: $f : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times U \times V \to \mathbb{R}$, continuous; Lipschitz in (x,y,z), uniformly w.r.t (t,u,v).

Under the above assumptions: existence and uniqueness of the solution of BSDE (2): $(Y^{t,\zeta;u,v}, Z^{t,\zeta;u,v}) \in \mathcal{S}^2_{\mathbb{F}}(t,T;\mathbb{R}) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^d)$.

From standard estimates for BSDEs using the corresponding results for the controlled stochastic system: $\exists C \in \mathbb{R}_+$ independent of $t, u, v, \zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, s.t.

$$|Y_t^{t,\zeta;u,v} - Y_t^{t,\zeta';u,v}| \leq C|\zeta - \zeta'|, \quad |Y_t^{t,\zeta;u,v}| \leq C(1+|\zeta|),\ P\text{-a.s.}$$

Let $t \in [0,T]$, $\zeta = x \in R^n$ - deterministic initial data; $u \in \mathcal{U}, v \in \mathcal{V}$; associated Cost functional for the game over the time interval $[t,T]$:

$J(t,x;u,v) := Y_t^{t,x;u,v}\big(\in L^2(\Omega, \mathcal{F}_t, P)\big)$.

Remark 1: (i) If $f \equiv 0 : J(t,x;u,v) = E[\Phi(X_T^{t,x;u,v})|\mathcal{F}_t]$;

(ii) If f doesn't depend on (y, z):

$$J(t,x;u,v) = E\left[\Phi(X_T^{t,x;u,v}) + \int_t^T f(s, X_s^{t,x;u,v}, u_s, v_s)ds|\mathcal{F}_t\right].$$

Which kind of game shall we study?

Objective of Player I: maximization of $J(t,x,u,v)$;

Objective of Player II: minimization of $J(t,x,u,v)$;

Game "Control against Control"?

• In general no value of the game, i.e., the result of the game depends on which player begins, and this even if Isaacs' condition is fulfilled (condition is given later).

• Games "Control against Control" with value if, for $n = d$: the matrix $\sigma(x)$ is invertible and independent of (u,v): In this context Isaacs' conditions guaranties the existence of saddle point feedback Nash equilibrium controls (S.HAMADENE, J.-P.LEPELTIER, S.PENG).

More general situations can be treated with:

Game "Strategy against Control":

This concept had been developed in the deterministic differential game theory (A.FRIEDMAN, W.H.FLEMING,..) and was translated later by W.H.FLEMING, P.E.SOUGANIDIS (1989) to the theory of stochastic differential games.

Here: a generalization of the concept of W.H.FLEMING, P.E.SOUGANIDIS (1989).

In recent papers by P.Cardaliaguet, C.Rainer:

Game "NSD against NSD":

NSD - Nonanticipative Strategy with Delay; Advantage/beauty of this concept: "symmetry" between both players.

Here: Game "Strategy against Control":

Admissible controls, admissible strategies

Definition 1: (*admissible controls* for a game over the time interval $[t,T]$)

- For Player I: $\mathcal{U}_{t,T} =: L^0_{\mathbb{F}}(t,T;U)$;
- for Player II: $\mathcal{V}_{t,T} =: L^0_{\mathbb{F}}(t,T;V)$.

Notice: In difference to the concept by FLEMING, SOUGANIDIS, the controls $u \in \mathcal{U}_{t,s}, v \in \mathcal{V}_{t,s}$ are not supposed to be independent of $\mathcal{F}_t$.

Definition 2: (*admissible strategies* for a game over the time interval $[t,T]$)

- For Player II: $\beta : \mathcal{U}_{t,T} \longrightarrow \mathcal{V}_{t,T}$ non anticipating, i.e.,

for any $\mathbb{F}-$ stopping time $S : \Omega \to [t,T]$ and any admissible controls $u_1,\, u_2 \in \mathcal{U}_{t,T}$

($u_1 = u_2$ dsdP-a.e. on $[\![t,S]\!] \Longrightarrow \beta(u_1) = \beta(u_2)$ dsdP-a.e. on $[\![t,S]\!]$).

$\mathcal{B}_{t,T} := \{\beta : \mathcal{U}_{t,T} \to \mathcal{V}_{t,T} | \beta \text{ is nonanticipating}\}$.

Analogously we introduce

• for Player I: $\mathcal{A}_{t,T} := \{\alpha : \mathcal{V}_{t,T} \to \mathcal{U}_{t,T} \,|\, \alpha \text{ is nonanticipating}\}$.

Value Functions:

Notice: For $J(t,x,u,v) := Y_t^{t,x,u,v}$:

i) $Y_t^{t,\zeta,u,v} = J(t,\zeta,u,v)\left(:= J(t,x,u,v)\big|_{x=\zeta}\right)$, P-a.s., for all $\zeta \in L^2(\mathcal{F}_t;\mathbb{R}^n)$;

ii) $J(t,\zeta,u,v) \in L^2(\Omega,\mathcal{F}_t,P)$ and:

$$|J(t,\zeta,u,v) - J(t,\zeta',u,v)| \le C|\zeta - \zeta'|,\ |J(t,\zeta,u,v)| \le C(1+|\zeta|),$$

P-a.s., for all $\zeta,\zeta' \in L^2(\mathcal{F}_t;\mathbb{R}^n)$, $(t,u,v) \in [0,T] \times \mathcal{U} \times \mathcal{V}$;

The above estimates for $J(t,x,u,v)$ allow to introduce:

• Lower Value Function:

$$W(t,x) := \text{essinf}_{\beta \in \mathcal{B}_{t,T}} \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t,x;u,\beta(u));$$

• Upper Value Function:

$$U(t,x) := \text{esssup}_{\alpha \in \mathcal{A}_{t,T}} \text{essinf}_{v \in \mathcal{V}_{t,T}} J(t,x;\alpha(v),v).$$

Remarks. • Justification of the names “upper” and “lower” value functions: later we will see $W \le U$; the proof is far from being obvious and uses the comparison principle for the associated Bellman-Isaacs equations, it will be given later.

• The esssup, essinf should be understood as ones w.r.t. a uniformly bounded, indexed family of $\mathcal{F}_t$-measurable r.v.; see: Dunford/Schwartz (1957).

Although $W,U \in L^\infty(\Omega,\mathcal{F}_t,P)$ are a priori r.v., we have the following crucial result:

Proposition 1: $W(t,x) = E[W(t,x)]$, $U(t,x) = E[U(t,x)]$, $(t,x) \in [0,T]\times\mathbb{R}^n$, i.e., W and U admit a deterministic version (with which we identify the both functions from now on).

From the estimates for $J(t,x;u,v) = Y_t^{t,x;u,v}$:

Corollary. $W,U : [0,T]\times\mathbb{R}^n \longrightarrow \mathbb{R}$ are such that

$$|W(t,x)-W(t,x')| + |U(t,x)-U(t,x')| \le C|x-x'|,$$

$$|W(t,x)| + |U(t,x)| \le C(1+|x|), \text{ for all } t\in[0,T],\ x,x'\in\mathbb{R}^n.$$

Some Remarks preceding the proof of the proposition.

1) Concept of W.H.FLEMING, P.E.SOUGANIDIS (1989):

their running cost $f(s,x,y,z)$ doesn't depend on (y,z), i.e., their cost functional is the classical one:

$$J(t,x;u,v) = E\left[\Phi(X_T^{t,x;u,v}) + \int_t^T f(s,X_s^{t,x;u,v},u_s,v_s)ds \mid \mathcal{F}_t\right]$$

More essential:

- *admissible controls*: $u \in \mathcal{U}_{t,T}^t := L^0_{\mathbb{F}^t}(t,T;U)$, $v \in \mathcal{V}_{t,T}$: $\mathcal{V}_{t,T}^t := L^0_{\mathbb{F}^t}(t,T;V)$ are independent of $\mathcal{F}_t$:
$$\mathbb{F}^t = (\mathcal{F}_s^t)_{s\in[t,T]},\ \mathcal{F}_s^t := \sigma\{B_r - B_t, r\in[t,s]\} \vee \mathcal{N}_P,\ s\in[t,T];$$
- *admissible strategies*: $\mathcal{B}_{t,T}^t$ - the set of all non anticipating mappings $\beta : \mathcal{U}_{t,T}^t \longrightarrow \mathcal{V}_{t,T}^t$; analogous definition of $\mathcal{A}_{t,T}^t$.

Their cost functional

$$J(t,x;u,v) := E\left[\Phi(X_T^{t,x,u,v}) + \int_t^T f(s, X_s^{t,x,u,v}, u_s, v_s) \,|\, \mathcal{F}_t\right]$$

$$= E\left[\Phi(X_T^{t,x,u,v}) + \int_t^T f(s, X_s^{t,x,u,v}, u_s, v_s)\right]$$

is automatically deterministic, and so are their upper and lower value functions:

$$\overline{W}(t,x) := \inf_{\beta\in\mathcal{B}_{t,T}} \sup_{u\in\mathcal{U}_{t,T}} J(t,x;u,\beta(u)), \overline{U}(t,x) := \inf_{\alpha\in\mathcal{A}_{t,T}} \inf_{v\in\mathcal{V}_{t,T}} J(t,x;\alpha(v),v).$$

We will see: if $f(s,x,y,z,u,v) = f(s,x,u,v)$ then $W = \overline{W}$, $U = \overline{U}$.

Our approach in comparison with theirs:

- Proof that W, U are deterministic is not evident, but after:
- Straight forward approach without approximation by discrete schemes, without further technical notions (like π-controls, r-strategies), without using the Bellman-Isaacs equation for proving the DPP:

- Direct deduction of the DPP from the definition of W, U (with the help of Peng's notion of backward semigroups);

- Direct deduction of the Bellman-Isaacs equations for W, U from the DPP (with the help of a scheme of 3 BSDEs, the so-called Peng's BSDE method developed by him for control problems);

2) Proof that W is deterministic for control problems (see, e.g., S.Peng, 1997):

$U \subset R^M$ compact subset; σ, b, f depend only on one control;

$$W(t,x) := \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t,x,u)$$

Approximation de $u \in \mathcal{U}_{t,T}$ in $L^2(\Omega \times [0,T]; U)$ by controls of the form $u^0 = \sum_{i=1}^n I_{A_i} u^i, (u^i \in \mathcal{U}_{t,T}^t, (A_i)_{1 \le i \le n} \subset \mathcal{F}_t$ partition of Ω);
and L^2-continuity of $(u \to J(t,x;u))$ gives

$$W(t,x) := \text{esssup}_{u^0} J(t,x,u),$$

where

$J(t,x;u^0) = \sum_{i=1}^{N} I_{A_i} J(t,x,u^i) \leq \sup_{1\leq i\leq N} J(t,x,u^i)$ - deterministic.

This argument doesn't work for SDG because of the presence of the strategies.

New approach for the proof that W is deterministic has been needed; even continuity of the coefficients in (u,v) is not needed anymore.

Proof that the lower value function is deterministic (analogous proof for upper value function); for simplicity: $f = 0$, i.e.,

$$J(t,x;u,v) = E\left[\Phi(X_T^{t,x;u,v}) \mid \mathcal{F}_t\right]$$

Main tool of the proof is a Girsanov transformation argument:

$\Omega = C_0([0,T];\mathbb{R}^d)$;

$B_t(\omega) = \omega(s), s \in [0,T], \omega \in \Omega$, is the coordinate process.

Let $H_t = \{h \in L_0'^2([0,T];\mathbb{R}^d) \mid h(s) = h(t),\ s \in [t,T]\}$;

for $h \in H_t$ we define:

$\tau_h(\omega) := \omega + h$, $\omega \in \Omega$;

$\tau_h : \Omega \to \Omega$ bijection; $\tau_h^{-1} = \tau_{-h}$;

$$\frac{dP_{\tau_h}}{dP} = L_h := \exp\{\int_0^t h'_s dB_s - \frac{1}{2}\int_0^t |h'_s|^2 ds\}.$$

Observe: since

$$dX_s^{t,x;u,v} \circ \tau_h = \sigma(X_s^{t,x;u,v} \circ \tau_h, u_s(\tau_h), v_s(\tau_h))dB_s,\ s \in [t,T],$$

we have $X_s^{t,x;u,v} \circ \tau_h = X_s^{t,x;u(\tau_h),v(\tau_h)}$, $s \in [t,T], u \in \mathcal{U}_{t,T}, v \in \mathcal{V}_{t,T}$.
Consequently,

$$J(t,x;u,v) \circ \tau_h = E\left[\Phi(X_T^{t,x;u,v} \circ \tau_h) \mid \mathcal{F}_t\right]$$
$$= E\left[\Phi(X_T^{t,x;u(\tau_h),v(\tau_h)}) \mid \mathcal{F}_t\right] = J(t,x;u(\tau_h),v(\tau_h)),\ P\text{-a.s.}$$

For $\beta \in \mathcal{B}_{t,T}$: $\beta_h(u) := \beta(u(\tau_h)) \circ \tau_{-h}$, $u \in \mathcal{U}_{t,T}$. Then $\beta_h \in \mathcal{B}_{t,T}$ and $(u \to u(\tau_h)) : \mathcal{U}_{t,T} \to \mathcal{U}_{t,T}$, $(\beta \to \beta_h) : \mathcal{B}_{t,T} \to \mathcal{B}_{t,T}$ are bijections.

Hence,

$W(t,x)\circ\tau_h = \text{essinf}_{\beta\in\mathcal{B}_{t,T}}\text{esssup}_{u\in\mathcal{U}_{t,T}}\,(J(t,x;u,\beta(u))\circ\tau_h)$

$= \text{essinf}_{\beta\in\mathcal{B}_{t,T}}\text{esssup}_{u\in\mathcal{U}_{t,T}} J(t,x;u(\tau_h),\beta(u)\circ\tau_h)$

$= \text{essinf}_{\beta\in\mathcal{B}_{t,T}}\text{esssup}_{u\in\mathcal{U}_{t,T}} J(t,x;u(\tau_h),\beta_h(u(\tau_h)))$

$= \text{essinf}_{\beta\in\mathcal{B}_{t,T}}\text{esssup}_{u\in\mathcal{U}_{t,T}} J(t,x;u,\beta_h(u)) = W(t,x)$, P-a.s.

Lemma: Let $\zeta\in L^0(\Omega,\mathcal{F}_t,P)$ be s.t. $\zeta\circ\tau_h=\zeta$, P-a.s., for all $h\in H_t$. Then there is a real $C\in\mathbb{R}$ s.t. $\zeta=C$, P-a.s. •

Dynamic Programming Principle (DPP)

Some Preparation: *Stochastic Backward Semigroup*, S.Peng,1997:

Given
$(t,\zeta)\in[0,T]\times L^2(\Omega,\mathcal{F}_t,P;\mathbb{R}^n),\ \delta\in(0,T-t],\ u\in\mathcal{U}_{t,t+\delta},\ v\in\mathcal{V}_{t,t+\delta},$
we put

$$G^{t,x;u,v}_{s,t+\delta}[\eta]:=\tilde{Y}_s,\ s\in[t,t+\delta],\ \eta\in L^2(\Omega,\mathcal{F}_{t+\delta},P),$$

where:

$$\left\{\begin{array}{rcl} d\tilde{Y}_s & = & -f(s,X^{t,x;u,v}_s,\tilde{Y}_s,\tilde{Z}_s,u_s,v_s)ds+\tilde{Z}^{t,x;u,v}_s dB_s, \quad \in[t,t+\delta], \\ \tilde{Y}_{t+\delta} & = & \eta; \end{array}\right.$$

$X^{t,x;u,v}$ is the solution of our doubly controlled forward SDE.

Remark:

(i) (*The semigroup property*) For $0\le t\le s\le s'\le t+\delta\le T$,

$$G^{t,x;u,v}_{s,s'}[G^{t,x;u,v}_{s',t+\delta}[\eta]]=G^{t,x;u,v}_{s,t+\delta}[\eta].$$

(ii) (*The classical case*) If f doesn't depend on (y,z) we have the classical case of conditional expectation:

$$G_{t,t+\delta}^{t,x;u,v}[\eta] = E\left[\eta + \int_t^{t+\delta} f(s, X_s^{t,x;u,v}, u_s, v_s)ds|\mathcal{F}_t\right], \text{ P-a.s.}$$

Taking now $\eta = W(t+\delta, X_{t+\delta}^{t,x;u,v})$ (resp., $U(t+\delta, X_{t+\delta}^{t,x;u,v})$) it becomes clear from the classical DPP from control problems that our DPP shall write as follows:

Theorem 2 (DPP): For any $0 \le t < t+\delta \le T$, $x \in \mathbb{R}^n$, P-a.s.,

$$W(t,x) = \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \operatorname*{esssup}_{u \in \mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{t,x;u,\beta(u)}[W(t+\delta, X_{t+\delta}^{t,x;u,\beta(u)})];$$

$$U(t,x) = \operatorname*{esssup}_{\alpha \in \mathcal{A}_{t,t+\delta}} \operatorname*{essinf}_{v \in \mathcal{V}_{t,t+\delta}} G_{t,t+\delta}^{t,x;\alpha(v),v}[U(t+\delta, X_{t+\delta}^{t,x;\alpha(v),v})].$$

Remark: If $f(x,y,z,u,v)$ is independent of (y,z) the above DPP writes:

$W(t,x) =$

$$\operatorname*{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \operatorname*{esssup}_{u \in \mathcal{U}_{t,t+\delta}} E[W(t+\delta, X_{t+\delta}^{t,x;u,\beta(u)}) + \int_t^{t+\delta} f(s, X_s^{t,x;u,\beta(u)}, u_s, v_s)ds|\mathcal{F}_t]$$

analogous for $U(t,x)$.

Proof of DPP: Direct (but rather lengthy) calculus.

With the help of the DPP and BSDE standard estimates we can prove:

Theorem 3. $W(.,x)$ and $U(.,x)$ are $\frac{1}{2}-$Hölder continuous, for all $x \in \mathbb{R}^n$: There is some $C \in \mathbb{R}_+$ such that, for every $x \in \mathbf{R}^n$, $t, t' \in [0,T]$,

$$|W(t,x) - W(t',x)| + |U(t,x) - U(t',x)| \leq C(1+|x|)|t-t'|^{\frac{1}{2}}.$$

Bellman-Isaacs equations. Existence theorem.

We consider the Hamiltonian

$$H(t,x,y,p,S,u,v)$$
$$:= \frac{1}{2}tr(\sigma\sigma^T(t,x,u,v)S) + b(t,x,u,v).p + f(t,x,y,p.\sigma(t,x,u,v),u,v),$$
$$(t,x,y,p,S,u,v) \in [0,T]\times\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n\times\mathbf{S}^n\times U\times V.$$
$$H^-(t,x,y,p,S) := \sup_{u\in U}\inf_{v\in V} H(t,x,y,p,S,u,v);$$
$$H^+(t,x,y,p,S) := \inf_{v\in V}\sup_{u\in U} H(t,x,y,p,S,u,v).$$

Then, in viscosity sense, we have the following Bellman-Isaacs equations:

$$\partial_t W(t,x) + H^-(t,x,W,DW,D^2W) = 0,\, W(T,x) = \Phi(x), \text{ Eq.}(H^-)$$

and

$$\partial_t U(t,x) + H^+(t,x,U,DU,D^2U) = 0,\, U(T,x) = \Phi(x). \text{ Eq.}(H^+)$$

More precisely,

Theorem 4: $W \in C_\ell([0,T]\times\mathbf{R}^n)$ is a viscosity solution of Eq.(H^-), and $U \in C_\ell([0,T]\times\mathbf{R}^n)$ is one of equation Eq.(H^+).

We come after back to the proof of the existence theorem.

Theorem 5 (Comparison Principle): Let $u_1 \in$USC$([0,T]\times\mathbb{R}^n)$ be a viscosity subsolution and $u_2 \in$LSC$([0,T]\times\mathbb{R}^n)$ a viscosity supersolution of Eq.(H^-) (of Eq.(H^-), resp.). Moreover, u_1, u_2 are supposed to belong to the class of functions V with the following growth condition:

$\exists A>0$ such that, uniformly in $t\in[0,T]$,

$$V(t,x)\exp\{-A[\ln|x|]^2\}\left(=\frac{V(t,x)}{|x|^{A\ln|x|}}\right)\longrightarrow 0 \text{ as } |x|\rightarrow+\infty.$$

Then $u_1 \leq u_2$, on $[0,T]\times\mathbb{R}^n$.

Corollary. Let u_1 and u_2 be continuous viscosity solutions of Eq.(H^-) (resp., of Eq.(H^+)). Moreover, we suppose that both functions satisfy the above growth condition. Then $u_1 = u_2$, on $[0,T]\times\mathbb{R}^n$.

<u>Remarks 1</u>: Barles, Buckdahn, Pardoux (1997) proved that this growth condition is the optimal one for the uniqueness of the (viscosity) solution of the heat equation.

Remarks 2: • $W \in C_\ell([0,T]\times\mathbb{R}^n)$ (resp., $U \in C_\ell([0,T]\times\mathbb{R}^n)$) is the unique viscosity solution of Eq.(H^-) (resp., Eq.(H^+)) in the class of continuous functions with the above growth condition, and so in particular in $C_p([0,T]\times\mathbb{R}^n)$.

• Notice that $H^- \le H^+$; consequently, W is a viscosity subsolution of Eq.(H^+), and from the comparison principle: $W \le U$. This justifies the name "*lower value function*" for W and "*upper value function*" for U.

• If the **Isaacs' condition** holds: $H^- = H^+$ on $[0,T]\times\mathbb{R}^n\times\mathbf{R}\times\mathbf{R}^n\times\mathbf{S}^n$, then the equations (H^-) and (H^+) are the same, and from the uniqueness of the viscosity solution in $C_p([0,T]\times\mathbb{R}^n)$: $W = U$. One says **the game has a value**.

• For $f(s,x,y,z,u,v) = f(s,x,u,v)$, W.H.FLEMING, P.E.SOUGANIDIS have the same Bellman-Isaacs equations as we have got. From the uniqueness of the viscosity solutions in $C_p([0,T]\times\mathbb{R}^n)$:

$$\overline{W}(t,x)\big(:= \inf_{\beta\in\mathcal{B}_{t,T}^t}\sup_{u\in\mathcal{U}_{t,T}^t} J(t,x;u,\beta(u))\big) = W(t,x);$$

$$\overline{U}(t,x)\big(:= \inf_{\alpha\in\mathcal{A}_{t,T}^t}\inf_{v\in\mathcal{V}_{t,T}^t} J(t,x;\alpha(v),v)\big) = U(t,x).$$

Sketch of the proof of the existence theorem:

We prove that W is a continuous viscosity solution of the PDE

$$\frac{\partial W}{\partial t}(t,x) + H^-(t,x,W,DW,D^2W) = 0, W(T,x) = \Phi(x), \qquad (3)$$

with

$$H^-(t,x,y,p,S) := \sup_{u\in U}\inf_{v\in V} H(x,y,p,S,u,v),$$
$$H(x,y,p,S,u,v) := \frac{1}{2}tr(\sigma\sigma^T(x,u,v)S) + f(x,y,p.\sigma(x,u,v),u,v),$$

$(x,y,p,S,u,v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \times U \times V$(for shortness: $b = 0$; coefficients don't depend on time s).

Let $\varphi \in C^3_{\ell,b}([0,T] \times \mathbb{R}^n)$ be an arbitrary but fixed test function. We define:

$$F(s,x,y,z,u,v) := \frac{\partial}{\partial s}\varphi(s,x) + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(x,u,v)D^2\varphi(s,x))$$
$$+ f(s,x,y+\varphi(s,x)), z + D\varphi(s,x)\sigma(x,u,v),u,v).$$

Notice:

$$\frac{\partial}{\partial t}\varphi(t,x)+H^{-}(t,x,(\varphi,D\varphi,D^2\varphi)(t,x)) = \sup_{u\in U}\inf_{v\in V}F(t,x,0,0,u,v).$$

So we have to prove that if $W-\varphi\le$ (resp., $\ge$)$=(W-\varphi)(t,x)=0$ then

$$\sup_{u\in U}\inf_{v\in V}F(t,x,0,0,u,v)\ge 0\ (\longrightarrow \text{subsolution})$$

$$(\text{resp., } \sup_{u\in U}\inf_{v\in V}F(t,x,0,0,u,v)\le 0\ (\longrightarrow \text{supersolution})).$$

Peng's BSDE method: *"Approximating BSDEs"*

1st BSDE: For $0<\delta\le T-t$, $u\in\mathcal{U}_{t,t+\delta}$, $v\in\mathcal{V}_{t,t+\delta}$:

$$dY_s^{1,u,v,\delta}=-F(s,X_s^{t,x,u,v},Y_s^{1,u,v,\delta},Z_s^{1,u,v,\delta},u_s,v_s)ds+Z_s^{1,u,v,\delta}dB_s,$$

$$Y_{t+\delta}^{1,u,v,\delta}=0.$$

Notice: $Y_s^{1,u,v,\delta}=G_{s,t+\delta}^{t,x,u,v}[\varphi(t+\delta,X_{t+\delta}^{t,x,u,v})]-\varphi(s,X_s^{t,x,u,v})$, $s\in[t,t+\delta]$, P-a.s.

The 1st BSDE translates the DPP into BSDE. Our objective: To approximate the 1st BSDE by the 2nd BSDE, and the 2nd BSDE by a deterministic ordinary differential equation with terminal condition.

2nd BSDE: For $0 < \delta \leq T - t$ small, $u \in \mathcal{U}_{t,t+\delta}, v \in \mathcal{V}_{t,t+\delta}$:

$dY_s^{2,u,v,\delta} = -F(s,x,Y_s^{2,u,v,\delta}, Z_s^{2,u,v,\delta}, u_s, v_s)ds + Z_s^{2,u,v,\delta} dB_s,\ s \in [t, t+\delta],$

$Y_{t+\delta}^{2,u,v,\delta} = 0.$

Notice: $\exists C \in \mathbb{R}_+$ s.t., for all $\delta > 0$, $u \in \mathcal{U}_{t,t+\delta}, v \in \mathcal{V}_{t,t+\delta}$:

$$|Y_t^{1,u,v,\delta} - Y_t^{2,u,v,\delta}| \leq C\delta^{3/2},\ P\text{-a.s.}$$

Let $F_0(s,x,y,z) = \sup_{u \in U} \inf_{v \in V} F(s,x,y,z,u,v)$.

3rd BSDE: For $0 < \delta \leq T - t$ small:

$dY_s^{0,\delta} = -F_0(s,x,Y_s^{0,\delta},0)ds\big(+0dB_s\big),\ s \in [t,t+\delta],$

$Y_{t+\delta}^{0,\delta} = 0.$

Notice: $\text{esssup}_{u \in \mathcal{U}_{t,t+\delta}} \text{essinf}_{v \in \mathcal{V}_{t,t+\delta}} Y_t^{2,u,v,\delta} = Y_t^{0,\delta}$.

These 3 BSDEs allow to prove that W is sub-and supersolution; here: proof that W is a supersolution:

Let $0=(W-\varphi)(t,x)\leq W-\varphi$. Then $\varphi\leq W$ and from DPP

$$
\begin{aligned}
0 &= W(t,x)-\varphi(t,x) \\
&= \operatorname{essinf}_{\beta\in\mathcal{B}_{t,T}}\operatorname{esssup}_{u\in\mathcal{U}_{t,T}}\left(G_{t,t+\delta}^{t,x,u,\beta(u)}(W(t+\delta,X_{t+\delta}^{t,x;u,\beta(u)}))-\varphi(t,x)\right) \\
&\geq \operatorname{essinf}_{\beta\in\mathcal{B}_{t,T}}\operatorname{esssup}_{u\in\mathcal{U}_{t,T}}\left((G_{t,t+\delta}^{t,x,u,\beta(u)}(\varphi(t+\delta,X_{t+\delta}^{t,x;u,\beta(u)}))-\varphi(t,x)\right) \\
&= \operatorname{essinf}_{\beta\in\mathcal{B}_{t,T}}\operatorname{esssup}_{u\in\mathcal{U}_{t,T}}Y_t^{1,\delta,u,\beta(u)} \\
&\geq \operatorname{essinf}_{\beta\in\mathcal{B}_{t,T}}\operatorname{esssup}_{u\in\mathcal{U}_{t,T}}Y_t^{2,\delta,u,\beta(u)}-C\delta^{3/2} \\
&\geq \operatorname{essinf}_{\beta\in\mathcal{B}_{t,T}}\left(\operatorname{esssup}_{u\in\mathcal{U}_{t,T}}\operatorname{essinf}_{v\in\mathcal{V}_{t,T}}Y_t^{2,\delta,u,v}\right)-C\delta^{3/2} \\
&= Y_t^{0,\delta}-C\delta^{3/2}, \text{ from where}
\end{aligned}
$$

$$
\begin{aligned}
&C\sqrt{\delta}\geq\frac{1}{\delta}Y_t^{0,\delta}=\frac{1}{\delta}\int_t^{t+\delta}F_0(s,x,Y_s^{0,\delta},0)ds \\
&\longrightarrow F_0(t,x,0,0)=\sup_{u\in U}\inf_{v\in V}F(s,x,0,0,u,v).
\end{aligned}
$$

Consequently, W is a supersolution of Eq.(H^-).

Some Remarks on Extensions

Above: cost functional $J(t,x;u,v) = Y_t^{t,x;u,v}$ is defined through the solution of a BSDE.

Extensions:
i) $J(t,x;u,v) = Y_t^{t,x;u,v}$ is defined through the solution of a BSDE reflected at one barrier;
ii) $J(t,x;u,v) = Y_t^{t,x;u,v}$ is defined through the solution of a reflected BSDE with two barriers.

Some Remarks on i): BSDEs with reflection (RBSDE) at one barrier, El Karoui, Kapoudjian, Pardoux, Peng, Quenez (1997):

For $(t,\zeta) \in [0,T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, $(u,v) \in \mathcal{U}_{t,T} \times \in \mathcal{V}_{t,T}$

$dY_s^{t,\zeta;u,v} = -f(s, X_s^{t,\zeta;u,v}, Y_s^{t,\zeta;u,v}, Z_s^{t,\zeta;u,v}, u_s, v_s)ds + Z_s^{t,\zeta;u,v} dB_s - dK_s^{t,\zeta;u,v}$,
$Y_s^{t,\zeta;u,v} \geq h(s, X_s^{t,\zeta;u,v})$, $(Y_s^{t,\zeta;u,v} - h(s, X_s^{t,\zeta;u,v}))dK_s^{t,\zeta;u,v} = 0$,
$K^{t,\zeta;u,v}$ continuous increasing process, $K_t^{t,\zeta;u,v} = 0$.

where $h : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is continuous, Lipschitz in x, unif. in t; compatibility assumption: $\Phi(x) \geq h(T,x)$, $x \in \mathbb{R}^n$.

Main difficulties we had to solve:

• RBSDE standard estimates give only:

$$|Y_t^{t,\zeta;u,v} - Y^{t,\zeta';u,v}| \leq C(1+|\zeta|+|\zeta'|)^{1/2}|\zeta-\zeta'|^{1/2}$$

Improvement of this standard estimate for RBSDEs with one and also with two reflecting barriers:

$$|Y_t^{t,\zeta;u,v} - Y^{t,\zeta';u,v}| \leq C|\zeta-\zeta'|;$$

(this sharper inequality: crucial for the proof of the DPP and of the proof that the value functions are viscosity solutions of the associated Bellman-Isaacs equations with obstacle)

Proof: combines estimates for RBSDEs with the same barrier and comparison theorem for BSDEs.

• Proof of the DPP: definition of the backward stochastic semigroup with the help of the RBSDE imposes that the estimates inside the backward stochastic semigroup have to respect the barrier (this makes the proof for BSDEs with two reflecting barriers even more delicate)

• Proof that W is a viscosity solution of the Bellman-Isaacs equation with obstacle.
$\min\big(W(t,x)-h(t,x), -\partial_t W(t,x)-H^-(t,x,(W,DW,D^2W)(t,x)\big)=0,$
$W(T,x)=\Phi(x), x\in\mathbb{R}^n.$

Proof:

- Penalization method: ${}^mY_s^{t,\zeta;u,v}\uparrow Y_s^{t,\zeta;u,v}$; $J_m(t,x;u,v):={}^m Y_t^{t,\zeta;u,v}$;
- $W_m(t,x):=\text{essinf}_{\beta\in\mathcal{B}_{t,T}}\text{esssup}_{u\in\mathcal{U}_{t,T}}J_m(t,x;u,v)$ unique viscosity solution of the associated penalized Bellman-Isaacs equation;
- $W_1\leq W_2\leq\cdots\leq W_m\uparrow\widetilde{W}\leq W$:
- $\widetilde{W}\in LSC_\ell([0,T]\times\mathbb{R}^n)$ supersolution of the above Bellman-Isaacs equation with obstacle;
- $W\in C_\ell([0,T]\times\mathbb{R}^n)$ subsolution of the above equation with obstacle: nontrivial combination of Peng's BSDE method with DPP, the approach differs from that for SDG without reflection;
- proof of the comparison principle; thus: $W\leq\widetilde{W}$.

Consequently: $W(=\widetilde{W})$ is the unique viscosity solution of the above equation with obstacle.