A simple continuous-time game with asymmetric information

Pierre Cardaliaguet, Catherine Rainer

Université de Bretagne Occidentale, Brest

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Outline

- Introduction
- Optimal strategies
- Non revealing set and optimal martingale
- The case $I = 2$

Introduction

Let $T > 0$ be a finite time horizon,
U and V some compact metric sets
and $I \in \mathbb{N}, I \geq 1$.

We investigate a continuous time game defined by a family of running costs :

For $i \in \{1, \dots, I\}$,
let $l_i : [0, T] \times U \times V \rightarrow \mathbb{R}$ bounded, continuous, Lipschitz in t uniformly in (u, v),

and $p = (p_1, \dots, p_I) \in \Delta(I)$ be a probability on $\{1, \dots, I\}$.

Introduction

The game is played in two steps :

- At initial time t the index i is chosen at random according to probability p.
  Index i is communicated to Player I only.
- Then
  - Player I tries to minimise the payoff

$$\sum_{i=1}^{I} p_i \int_t^T l_i(s, u_i(s), v(s))ds$$

  - Player II tries to maximise this payoff.

This is a continuous times version of the repeated game studied by Aumann and Maschler in 1995.

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Random strategies and associated payoff

Notations :

The set of random strategies for Player I (resp. Player II) is denoted by $\mathcal{A}_r(t)$ (resp. $\mathcal{B}_r(t)$).

Payoff

For $p \in \Delta(I)$ and $((\alpha_i), \beta) \in (\mathcal{A}_r(t))^I \times \mathcal{B}_r(t)$ we consider the following payoff :

$$\sum_{i=1}^{I} p_i \mathbf{E}_{\alpha_i, \beta} \left[\int_t^T l_i(s, \alpha_i, \beta) ds \right]$$

Hamiltonian, Isaacs' condition and value of the game

Assumption

We assume that Isaacs' condition holds :
For all $t \in [0,T]$ and $p \in \Delta(I)$:

$$H(t,p) := \inf_u \sup_v \left(\sum_{i=1}^{I} p_i l_i(t,u,v) \right) = \sup_v \inf_u \left(\sum_{i=1}^{I} p_i l_i(t,u,v) \right)$$

Theorem

Under Isaacs' condition, the game has a value

$$V(t,p) :=$$
$$= \inf_{(\alpha_i)\in(\mathcal{A}_r(t))^I} \sup_{\beta\in\mathcal{B}_r(t)} \left(\sum_{i=1}^{I} p_i \mathbf{E}_{\alpha_i,\beta}\left[\int_t^T l_i(s,\alpha_i,\beta)ds \right] \right)$$
$$= \sup_{\beta\in\mathcal{B}_r(t)} \inf_{(\alpha_i)\in(\mathcal{A}_r(t))^I} \left(\sum_{i=1}^{I} p_i \mathbf{E}_{\alpha_i,\beta}\left[\int_t^T l_i(s,\alpha_i,\beta)ds \right] \right) .$$

Theorem

V is the unique function that is bounded, continuous, Lipschitz with respect to t, convex in p and solution in viscosity sense of

$$\begin{cases} \min\left\{w_t + H(t,p), \lambda_{\min}\left(\frac{\partial^2 w}{\partial p^2}\right)\right\} = 0, \ (t,p) \in [0,T] \times \Delta(I), \\ w(T,p) = 0, \ p \in \Delta(I), \end{cases}$$

where, for any symmetric matrix A, $\lambda_{\min}(A)$ is the smallest eigenvalue of A.

Martingales

Definition

On a probability space $(\Omega, \mathcal{F}, P)$ sufficiently large, let $\mathcal{P}(t, p)$ be the set of all càdlàg martingales $(\mathbf{p}(s), s \in [t, T])$ with values in $\Delta(I)$ that satisfy :

$$E[\mathbf{p}(t-)] = p \; ; \; \mathbf{p}(T) \in \{e_1, \ldots, e_I\} \; P\text{-a.s.},$$

where $\{e_1, \ldots, e_I\}$ is the canonical basis of $\mathbb{R}^I$.

An equivalent formulation for the value function V

Theorem

- For all $(t, p) \in [0, T] \times \Delta(I)$,

$$V(t,p) = \inf_{\mathbf{p} \in \mathcal{P}(t,p)} E\left[\int_t^T H(s, \mathbf{p}(s))ds\right].$$

- The infimum is attained :
  up to a change of the probability space, we can find $\overline{\mathbf{p}} \in \mathcal{P}(t,p)$ such that

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Optimal strategies

- Let $u^* = u^*(t,p)$ be a Borel measurable selection of $\text{argmin}_{u \in U} \left(\max_{v \in V} \sum_{i=1}^{I} p_i l_i(t,u,v) \right)$.
- For $(t,p) \in [0,T] \times \Delta(I)$ fixed, let $\bar{\mathbf{p}}$ be optimal for $\min_{\mathbf{p} \in \mathcal{P}(t,p)} \mathbf{E}[\int_t^T H(s,\mathbf{p}(s))ds]$.
- For all $i \in \{1,\ldots,,I\}$, let $\bar{\mathbf{u}}_i(s) \stackrel{d}{=} u^*(s,\bar{\mathbf{p}}(s))|_{\{\bar{\mathbf{p}}(T)=e_i\}}$.

Theorem

$(\bar{\mathbf{u}}_1,\ldots,\bar{\mathbf{u}}_I)$ is optimal for $V(t,p)$:

$$V(t,p) = \sup_{\beta \in \mathcal{B}_r(t)} \sum_{i=1}^{I} E_{\bar{\mathbf{u}}_i,\beta(u_i)} \left[\int_t^T l_i(s,\bar{\mathbf{u}}_i(s),\beta(\bar{\mathbf{u}}_i)(s))ds \right].$$

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The non revealing set $\mathcal{H}$

For the simplicity of the talk,

we suppose from now on that $V \in C^{1,2}$.

Then V satisfies

$$\begin{cases} \min\left\{V_t + H(t,p), \lambda_{\min}(\frac{\partial^2 V}{\partial p^2})\right\} = 0, \ (t,p) \in [0,T] \times \Delta(I), \\ V(T,p) = 0, \ p \in \Delta(I), \end{cases}$$

We set $\mathcal{H} = \left\{(t,p) \in [0,T] \times \Delta(I) \mid V_t + H(t,p) = 0\right\}$.

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The optimal martingale

Theorem

Let $\bar{\mathbf{p}} \in \mathcal{P}(t, p)$ be optimal for $V(t, p)$.

Then, for all $s \in [t, T]$ P-a.s.,

- $(s, \bar{\mathbf{p}}(s)) \in \mathcal{H}$,
- $V(s, \bar{\mathbf{p}}(s)) - V(s, \bar{\mathbf{p}}(s-)) - \langle \frac{\partial V}{\partial p}(s, \bar{\mathbf{p}}(s-)), \bar{\mathbf{p}}(s) - \bar{\mathbf{p}}(s-) \rangle = 0.$

Verification Theorem

Theorem

Suppose that some martingale $\mathbf{p} \in \mathcal{P}(t,p)$ satisfies :

- P-a.s. $\forall s \in [t, T]$,
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- $\mathbf{p}$ is purely discontinuous.

Then $\mathbf{p}$ is optimal.

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Idea of the Proof : By Itô's formula,

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0 = \quad & E[V(T, \mathbf{p}(T))] \\
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$I = 2$

For $I = 2$, suppose that
there exist $h_1, h_2 : [0, T] \to [0, 1]$ continuous, $h_1 \leq h_2$,
h_1 decreasing and h_2 increasing such that

- $\text{Vex}H(t, p) = H(t, p) \iff p \in [0, h_1(t)] \cup [h_2(t), 1]$,
- $\frac{\partial^2 H}{\partial p^2}(t, p) > 0 \quad \forall (t, p)$ with $p \in [0, h_1(t)) \cup (h_2(t), 1]$.

$I = 2$

Proposition 1

Under the above assumptions,

- $V(t,p) = \int_t^T \text{Vex}H(s,p)ds \qquad \forall (t,p) \in [0,T] \times [0,1]$,
- $\mathcal{H} = \Big\{(t,p) \in [0,T] \times [0,1] \mid \ p \in [0, h_1(t)] \cup [h_2(t), 1]\Big\}$.

$I = 2$

Proposition 2

Under the above assumptions, there exists an unique optimal martingale $\bar{\mathbf{p}} \in \mathcal{P}(t, p)$ that satisfies

- $\bar{\mathbf{p}}$ is purely discontinuous,
- $\bar{\mathbf{p}}(s-) = p \qquad \forall s \in [t, t^*]$ P-a.s.,

  where $t^* = \inf\{s \geq t \mid p \in [h_1(t), h_2(t)]\}$,
- $\bar{\mathbf{p}}(s) \in \{h_1(s), h_2(s)\}, \qquad \forall s \in [t^*, T)$, P-a.s..