Repeated games and evolution equations in discrete and continuous time

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Dynamic games, Differential games III

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Definition

A zero-sum stochastic game is a 5-tuple (Ω, A, B, g, ρ) where :

- Ω is the set of states
- A (resp. B) is the action state of J_1 (resp. J_2)
- $g : A \times B \times \Omega \to \mathbb{R}$ is the payoff function, which will be assumed bounded.
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How the Game is played

An initial state ω_1 is given, known by each player.
During each stage i :

- the players observe the current state ω_i.
- According to the past history, J_1 (resp. J_2) choose a mixed strategy x_i in $\Delta(A)$ (resp. y_i in $\Delta(B)$).
- An action a_i of player 1 (resp. b_i of player 2) is drawn according to his mixed strategy x_i (resp. y_i).
- This gives the payoff at stage i $g_i = g(a_i, b_i, \omega_i)$.
- A new state ω_{i+1} is drawn according to $\rho(a_i, b_i, \omega_i)$.

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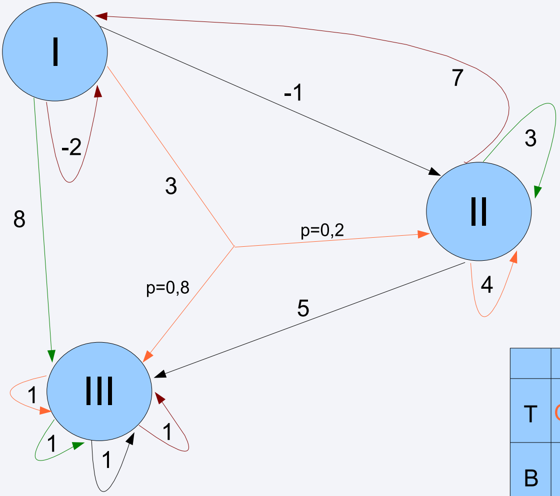
Example



	L	R
T	Orange	Green
B	Red	Black

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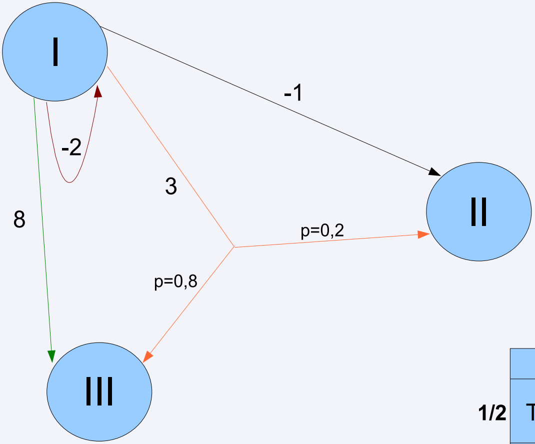

		1/4	3/4
		L	R
1/2	T	1/8	3/8
1/2	B	1/8	3/8

Payoff of the repeated game

There are several ways of evaluating a payoff for a given infinite history :

- $\frac{1}{n}\sum_{i=1}^{n} g_i$ is the payoff of the $n-$stage game
- $\lambda \sum_{i=1}^{+\infty}(1-\lambda)^{i-1} g_i$ is the payoff of the $\lambda-$discounted game.

If those games have a value for a given initial state ω, we denote them by $v_n(\omega)$ and $v_\lambda(\omega)$ respectively.
Thus v_n and v_λ are functions from Ω into $\mathbb{R}$.

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Asymptotic behavior

The main problem which arises is the study of the behavior of v_n when $n \to +\infty$ and of v_λ when $\lambda \to 0$. Does the limits exist, and are they the same ?

We know that the answers to both questions are positive in several cases :

- Finite stochastic games (Ω, A and B finite)
- Absorbing games
- Recursive games
- Games with incomplete information and standard signaling

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\begin{aligned}
\Psi(f)(\omega) &= Val_{\Delta(A)\times\Delta(B)} \left\{ g(x,y,\omega) + E_{\rho(x,y,\omega)}(f) \right\} \\
&= \sup_{x\in\Delta(A)} \inf_{y\in\Delta(B)} \left\{ g(x,y,\omega) + E_{\rho(x,y,\omega)}(f) \right\} \\
&= \inf_{y\in\Delta(B)} \sup_{x\in\Delta(A)} \left\{ g(x,y,\omega) + E_{\rho(x,y,\omega)}(f) \right\}.
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We will assume that Ψ is well defined by these equations and maps $\mathscr{F}$ into itself.

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The family of operators $\Phi(\alpha,\cdot)$

From Ψ we can define a family of operators $\Phi(\alpha,\cdot)$ for $\alpha\in]0,1]$ by the formula

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Properties of Ψ and $\Phi(\alpha,\cdot)$

The operator Ψ is topical : it satisfies the two following properties :

- Monotonicity $f_1 \le f_2 \implies \Psi(f_1) \le \Psi(f_2)$
- Homogeneity $c \in \mathbb{R} \implies \Psi(f+c) = \Psi(f) + c$

These two properties implies that Ψ is nonexpansive for the infinite norm

$$\|\Psi(f) - \Psi(g)\|_\infty \le \|f-g\|_\infty$$

and thus that $\Phi(\alpha,\cdot)$ is $1-\alpha$ contracting for the infinite norm

$$\|\Phi(\alpha,f) - \Phi(\alpha,g)\|_\infty \le (1-\alpha)\|f-g\|_\infty$$

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Recursive formulas

The utility of these operators lies in the fact that v_n and v_λ, providing that they are well defined, satisfies the following formulas :

$$v_n = \Phi\left(\frac{1}{n}, v_{n-1}\right) = \frac{\Psi^n(0)}{n}$$

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Generalization of framework

Let $(X, \|\|)$ be a Banach space, and let $\Psi : X \to X$ be an nonexpansive operator.

Let us define the family of contracting operators $\Phi(\alpha, \cdot) : X \to X$ by the formula

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and then let us define the elements v_n and v_λ of X by the formulas

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Questions

We now settle the three following questions :

- Does $\lim_{n \to +\infty} v_n$ exist ?
- Does $\lim_{\lambda \to 0} v_\lambda$ exist ?
- Are those two limits equal ?

Positive results

Besides the positive results already exposed in the section about stochastic games, we know that the three questions can be answered positively

- when Ψ is a topical function from $\mathbb{R}^2$ into itself
- and so in particular when Ψ is the Shapley operator of any 2 state game
- when Ψ is the Shapley operator of any 3 state compact game.

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Positive results

There are also positive results about the convergence of the norm of the values :

Theorem (Kohlberg Neyman)

It is always true that

$$\lim_{n\to+\infty} \|v_n\| = \lim_{\lambda\to 0} \|v_\lambda\| = \inf_{x\in X} \|\Psi(x) - x\|$$

Theorem (Gaubert Gunawardena)

Furthermore, if Ψ is the Shapley operator of a game, then

$$\lim_{n\to+\infty} \mathbf{t}(v_n) = \lim_{\lambda\to 0} \mathbf{t}(v_\lambda) = \inf_{x\in X} \mathbf{t}(\Psi(x) - x)$$

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Negative results

The answer to one of the three question may be negative, for example :

- if the norm of X^* isn't Fréchet-differentiable, then there is Ψ such that both v_n and v_λ diverge.
- In particular, there is a 3 state game (but with non-bounded payoff) such that neither v_n nor v_λ converge.
- There is a one-player game, with $\Omega = \mathbb{N}^2$ and $|A| = 2$ such that v_n and v_λ converge, but to two different limits.

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Summary table

If we put in ordinate the dimension of X (that is the number of states if we are in the case of a game), then we have the following answer :

	Nonexpansive	Shapley	Compact game	Finite game
1	Y	Y	Y	Y
2	N	Y	Y	Y
3	N	N	Y	Y
4+	N	N	?	Y
$+\infty$	N	N	N	N

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Evolution equation related to V_n

Let us denote $V_n = nv_n = \Psi^n(0) = \Psi(V_{n-1})$ and $A = I - \Psi$

We consider the differential equation

$$U(t) + U'(t) = \Psi(U(t)) \quad ; \quad U(0) = U_0 \in X. \tag{1}$$

that is

$$U'(t) = -A(U(t)) \quad ; \quad U(0) = U_0 \in X. \tag{2}$$

Notice that A is a maximal monotone operator. Equation A is usually studied in Hilbert space and assuming that $A^{-1}(0) \neq \emptyset$ but here this is not the case.

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Evolution equation related to v_n

Proposition

The solution of (1) satisfies

$$\|U(n) - V_n\| \leq \|U_0\| + \sqrt{n} \cdot \|\Psi(0)\|.$$

$$\left\| \frac{U(n)}{n} - v_n \right\| \to 0.$$

Corollary

Let $\tau(t) = t + \ln(1+t)$, and let u be the solution of the evolution equation

$$u(t) + u'(t) = \Phi\left(\frac{1}{2 + \tau^{-1}(t)}, u(t)\right).$$

Then $\|u(n) - v_n\| \to 0$.

Evolution equation related to v_n

Proposition

The solution of (1) satisfies

$$\|U(n) - V_n\| \le \|U_0\| + \sqrt{n} \cdot \|\Psi(0)\|.$$

$$\left\| \frac{U(n)}{n} - v_n \right\| \to 0.$$

Corollary

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Eulerian exponential formula

In addition to the traditional exponential formula for maximal monotone operators

$$U(t) = \lim_{n\to+\infty} \left(Id + \frac{t}{n}A\right)^{-n} (U_0)$$

we also get an "Eulerian" exponential formula

Proposition

$$\forall t \geq 0,\ U(t) = \lim_{n\to+\infty} \left(Id - \frac{t}{n}A\right)^{n} (U_0)$$

Eulerian Kobayashi

If λ_n is a sequence of reals in $]0,1]$, let us define W_n by

$$\frac{W_n - W_{n-1}}{\lambda_n} = -A(W_{n-1}).$$

$$W_n = (1-\lambda_n)W_{n-1} + \lambda_n \Psi(W_n)$$

We denote $\sigma_n = \sum_{i=1}^n \lambda_i \quad ; \quad \tau_n = \sum_{i=1}^n \lambda_i^2$.

Proposition

If W_n and $\widetilde{W}_n$ are defined from λ_n and $\widetilde{\lambda}_n$, then

$$\left\|W_k - \widetilde{W}_l\right\| \quad \leq \quad \|W_0\| + \|\widetilde{W}_0\| + \|\Psi(0)\| \sqrt{(\sigma_k - \widetilde{\sigma}_l)^2 + \tau_k + \widetilde{\tau}_l}$$

$$\frac{\|W_k - U(\sigma_k)\|}{\sigma_k} \quad \leq \quad \frac{\|W_0\| + \|U_0\|}{\sigma_k} + \frac{\|\Psi(0)\|}{\|\sqrt{\sigma_k}\|}.$$

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Table of contents

When λ is fixed

Proposition

When λ is fixed, the solution u of the evolution equation

$$u(t)+u'(t)=\Phi(\lambda,u(t)) \quad ; \quad u(0)=u_0\in X \tag{3}$$

satisfies

$$\lim_{t\to+\infty} u(t)=v_\lambda$$

Sketch of proof

Lemma

The solution of (3) satisfies $\|u(t) - v_\lambda\| \leq \frac{\|u'(t)\|}{\lambda}$.

Lemma

If f satisfies $\|f(t) + f'(t)\| \leq (1-\lambda(t))\|f(t)\|$, then

$$\|f(T)\| \leq \|f(0)\| e^{-\int_0^T \lambda(t)dt}.$$

Let us apply the second lemma to $f_h = \frac{u(t+h)-u(t)}{h}$, so that $\|f_h(t)\| \leq \|f_h(0)\| e^{-\lambda t}$. We then let h go to 0 and we use the first lemma.

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Non autonomous case

We are now interested in the equation of the type

$$u(t)+u'(t)=\Phi(\lambda(t),u(t)) \quad ; \quad u(0)=u_0\in X \tag{4}$$

where λ is a continuous function from $\mathbb{R}^+$ into $]0,1[$.

Proposition

If $\lambda \notin \mathscr{L}^1$, then the asymptotic behavior of the solution of (4) doesn't depend of u_0.

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Hypothesis on $\Phi(\cdot, x)$

From now on we make the following hypothesis :
$\exists C \in \mathbb{R},\ \forall (\lambda, \mu) \in]0,1[^2,\ \forall x \in X,$

$$\|\Phi(\lambda, x) - \Phi(\mu, x)\| \leq C|\lambda - \mu|(1 + \|x\|) \qquad (\mathscr{H})$$

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Consequences (I)

Proposition

Let λ and $\widetilde{\lambda}$ be two parametrization, and let u and $\widetilde{u}$ be the corresponding solutions of (4). If $\lambda \notin \mathscr{L}^1$, if u is bounded and if $\lambda(t) \sim \widetilde{\lambda}(t)$ then $\lim_{t \to +\infty} \|u(t) - \widetilde{u}(t)\| = 0$

Corollary

- *If $\lambda(t) \to \lambda_0 > 0$ then $u(t) \to v_{\lambda_0}$.*
- *If $\lambda(t) \sim \frac{1}{t}$ then $\|u(n) - v_n\| \to 0$.*

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If $\lim_{t \to +\infty} \frac{\lambda''(t)}{\lambda(t)\lambda'(t)} = 0$ *then the rate of convergence is in* $O\left(\frac{\lambda'(t)}{\lambda^2(t)}\right)$.

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If $\lambda(t) \sim \frac{1}{t^\alpha}$ *for an* $\alpha \in]0,1[$ *then* $\left\|u(t) - v_{\lambda(t)}\right\| \to 0$.
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Back to discrete time

For every λ_n sequence of numbers in $]0,1[$ let us define the sequence w_n of element of X by

$$w_n = \Phi(\lambda_n, w_{n-1})$$

Proposition

If $\lambda_n \to 0$ and $\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}} \to 0$, then $\|w_n - v_{\lambda_n}\| \to 0$

Corollary

If $\lambda_n \to 0$, $\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}} \to 0$, and if w_n converges, then v_λ converges to the same limit.

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Summary

- We can express the dynamic structure of stochastic games by the mean of a nonexpansive operator defined on a Banach space.
- The asymptotic behavior of values of those games is related to the asymptotic behavior of the solutions of certain evolution equations.

- Perspectives
  - What is the asymptotic behavior of those solutions ?
  - We haven't used the monotonicity of Ψ at all.
  - The derived game.

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