Nonlinear and infinite dimensional E.S.S and Wardrop equilibria: some new results and examples

Pierre Bernhard

I3S, University of Nice-Sophia Antipolis and CNRS
France

Rencontres de Roscoff, novembre 2008

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The classics: previous centuries

Seminal papers

John G Wardrop: “Some theoretical aspects of road trafic research”, *Proceedings of the Institution of Civil Engineers*, pp 325–378, 1952

John Maynard-Smith & G.R. Price: “The nature of animal conflict”, *Nature* **46**, pp 15–18, 1973.

P. Taylor & L. Jonker: “Evolutionary Stable Strategies and games dynamics”, *Mathematical Biosciences* **40**, pp 145–156, 1978

The classics: previous centuries

Volumes

John Maynard-Smith: *Evolution and the Theory of Games*, Cambridge University Press, 1982

Jörgen Weibull: *Evolutionary Game Theory*, MIT Press, 1995

Josef Hofbauer & Karl Sigmund: *Evolutionary Games and Replicator Dyamics*, Cambridge University Press, 1998

Some of the moderns

R. Cressman: *Evolutionary Dynamics and Games in Extensive form*, M.I.T. Press, 2003.

T.L. Vincent and J.S. Brown: *Evolutionary Game Theory, Natural Selection and Darwinian Dynamics*, Cambridge University Press, 2006.

Future

William Sandholm: *Population Games and Evolutionary Dynamics*, MIT Press, to appear p.s.b.n.

What is there in common

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Answer

How individual selfish behaviour of many identical individuals translates into population equilibrium.

Game of one individual against a population of identical individuals playing the same game for themselves.

Framework and notation

A large population of agents. Each has a choice of strategies $x \in X$.

$\Delta(X)$ is the set of positive measures of mass 1 (probabilities) over X.

For $A \subset X$, $n(A)$ the number of agents using a strategy $x \in A$,
$q(A) = n(A)/\int_X n(\mathrm{d}x)$ the *proportion* of the population using $x \in A$.
($q(A)$ is also the probability that an agent picked at random with a uniform probability over the population uses a strategy $x \in A$.)

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Three cases

- X finite, $X = \{x_1, x_2, \dots, x_n\}$, denote $q(x_i) = q_i$, $q \in \Sigma_n \subset \mathbb{R}^n$,
- $X \subset \mathbb{R}^n$, (hardly considered here), q is a measure over a continuum,
- X of infinite dimension (control). Two examples will be provided.

Strategies

"strategies" x may be phenotypes (Evolution), instinctive or learned behaviours (behavioural ecology), routing strategies (road engineering, routing in a communication network), trading strategies (stock market), etc.

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from

a *polymorphic* population where each agent uses always the same strategy, but collective mixed behaviour results from shares of the population choosing each strategy. (As explained bove)

Fitness and generating function

Hypothesis The *fitness* that gets an agent using a strategy x is a function $G(x,q)$ of x and the *distribution of stategies* q across the population.

The collective fitness of a sub-population using a distribution r within itself in a larger population of overall distribution q is

$$F(r,q) = \int_X G(x,q)r(\mathsf{d}x)\,.$$

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Linear case

An important sub-case is when $q \mapsto G(x,q)$ is linear (a math. expectation)

$$G(x,q) = \int_X H(x,y)q(\mathrm{d}y)\,, \quad F(r,q) = \iint_{X\times X} H(x,y)q(\mathrm{d}y)r(\mathrm{d}x)\,.$$

This is not necessary for many results.

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Nonlinear case

In the general case, $q \mapsto G(x,q)$ is nonlinear, we let

$$\mathsf{D}_2 G(x,q) = H(x,y,q)\,, \quad \mathsf{D}_2 G(x,q)\cdot r = \int_X H(x,y,q) r(\mathrm{d}y)\,.$$

Evolutionary stability

In a population with initial distribution p, a fraction ε mutates to q. The overall distribution is then

$$q_\varepsilon = p + \varepsilon(q - p)\,.$$

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This sub-population *invades* the original one if $F(q, q_\varepsilon) \geq F(p, q_\varepsilon)$. Hence the population is protected against invasion, or *evolutionarily stable*, if

$$\forall q \in \Delta(X) - \{p\}\,, \exists \varepsilon_0 > 0 : \forall \varepsilon \in (0, \varepsilon_0)\,, \quad F(q, q_\varepsilon) < F(p, q_\varepsilon)\,.$$

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The supremum of such ε_0's is called the *invasion barrier*.

Wardrop equilibrium

In the E.S.S. condition above, let $\varepsilon \to 0$. It comes

$$F(p,p) = \max_{q\in\Delta(X)} F(q,p)$$

(p,p) is a Nash point of the game $J_1(p,q) = F(p,q)$, $J_2(p,q) = F(q,p)$. Hence Von-Neumann's *equalization theorem* holds:

$$\forall x \in X\,,\; G(x,p) \le F(p,p)\,, \quad p(\{x \mid G(x,p) < F(p,p)\}) = 0\,. \qquad \text{(W)}$$

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Quote *"The journey times on all routes actually used are equal, and less than those that would be experienced by a single vehicle on any unused route [. . .] The first criterion is quite a likely one in practice [. . .] an equilibrium situation in which no driver can reduce his journey time by choosing a new route"* John Glen Wardrop, 1952

Linear case : second order condition

The Wardrop condition is only necessary. Let the set of *best responses* to p be $B(p) = \{r \mid F(r,p) = \max_q F(q,p) = F(p,p)\}$

Proposition In the linear case, a Wardrop equilibrum p is an E.S.S. iff

$$\forall q \in B(p)\,, \quad \langle (q-p), H(q-p)\rangle < 0\,.$$

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Let $X_1 = \{x \mid G(x,p) = F(p,p)\}$ and $X_2 = \text{support}(p) \subset X_1$. (W)
Let H_1 be the restriction of H to $X_1 \times X_1$ and H_2 to $X_2 \times X_2$.

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Let H_1 be the restriction of H to $X_1 \times X_1$ and H_2 to $X_2 \times X_2$.

Theorem In the linear case, a Wardrop equilibrium p is an E.S.S.
if the restriction of the quadratic form $\langle r, H_1 r\rangle$ to $r \in \mathbb{1}^\perp \subset \Delta(X_1)$ is negative definite,
and only if the restriction of the quadratic form $\langle r, H_2 r\rangle$ to $r \in \mathbb{1}^\perp \subset \Delta(X_2)$ is non-positive definite.

Finite linear case : a simple test

$$A = \begin{pmatrix} 1 & 4 & 5 \\ 7 & 3 & 9 \\ 4 & 6 & 2 \end{pmatrix}$$

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A passes the test:

$$\sigma(A) + \sigma(A)^t = \begin{pmatrix} -14 & 11 \\ 11 & -20 \end{pmatrix} < 0\,.$$

Nonlinear case : second order condition

$H(x,y,q) = \mathsf{D}_2 G(x,q)$. Let H_1 be its restriction to $X_1 \times X_1 \times \mathcal{M}(X_1)$, and similarily H_2 its restriction to $X_2 \times X_2 \times \mathcal{M}(X_2)$.

Definition A (finite) Wardrop equilibrium is *regular* if $q \mapsto H_1(q)$ is continuous at p and the restriction of $\langle r, H_1(p)r\rangle$ to $r \in \mathbb{1}^\perp \subset \mathcal{M}(X_1)$ is negative definite.

Theorem For a Wardrop equilibrium to be an ESS, it is

necessary that the restriction of the quadratic form $\langle r, H_2 r\rangle$ to $r \in \mathbb{1}^\perp \subset \mathcal{M}(X_2)$ be nonpositive definite,

sufficient in the finite case that it be regular.

Local Superiority

Definition A strategy distribution p is called *locally superior* or equivalently p is an *Evolutionarily Robust Strategy* (E.R.S.) (or a *Neighborhood Invading Strategy* N.I.S.) if there exists a neighborhood $\mathcal{N}$ of p such that

$$\forall q \in \mathcal{N} - \{p\}, \quad F(p,q) > F(q,q)$$

Easy result: E.R.S. $\Rightarrow$ E.S.S. (Place q_ε in above definition and use the linearity of F w.r.t. its first argument.)

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More difficult:

Theorem In the finite, linear or regular case, E.S.S. $\Rightarrow$ E.R.S.

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$$H = \begin{pmatrix} 1 & \pi \\ 2\pi & 0 \end{pmatrix} . \qquad \sigma(H) = 1 - 3\pi < 0 .$$

If $\pi = 1/2$, $p_1 = 1$ and $G_i(p) = 1$.
If $\pi = 2/3$, $p_1 = 2/3$ and $G_i(p) = 8/9 < 1$.

This is an instance of Braess'paradox, well known in the transportation literature.

Replicator dynamics

Let $n(x)$ be the number (or density) of individuals using strategy x.
Let $q(x) = n(x)/\int_X \mathrm{d}n(y)$ the strategy distribution.
Assume $G(x, q)$ is the *reproductive efficiency*. Then

Discrete generations Generation duration h

$$q(x, t + h) = q(x, t)\frac{1 + hG(x, q)}{1 + hF(q, q)}\,.$$

Continuous time The limit as $h \to 0$:

$$\dot{q}(x, q) = q[G(x, q) - F(q, q)]\,.$$

Stability of the replicator equation

Theorem

- Any limit point of the replicator dynamics is a Wardrop equilibrium,

- in finite dimension, E.S.S. are Lyapunov asymptotically stable. Its attraction basin contains a neighborhood of the relative interior of the lowest dimensional face of $\Delta(X)$ it lies on.

The stability proof uses the relative entropy of q to p as Lyapunov function. Its derivative is negative if p is an E.R.S. But we have no stability result of E.R.S. in the infinite case, because that function is not weakly continuous.

A population game

Lynxes and wolves

$L\backslash W$	$cow.$	$agr.$
$cow.$	$1-\lambda$ λ	1 0
$agr.$	0 $1-\mu$	$-\theta$ $1-\nu$

$$\lambda + \mu > 1 > \nu$$

$$\sigma^1 = \lambda + \mu - \nu\,, \quad p^2 = (1-\nu)/(\lambda+\mu-\nu)\,,$$
$$\sigma^2 = -\lambda - \theta\,, \qquad p^1 = \theta/(\lambda+\theta)\,.$$

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Draw case $\quad \lambda = \nu = 0{,}5, \quad \mu = 0{,}75, \quad \theta = 1{,}5.$

Population games and replicator dynamics

Replicator equation of the 2×2 case: mixed Nash equilibrium (p^1, p^2),

$$\dot{q}^k = \sigma^k q^k (1 - q^k)(q^\ell - p^\ell)\,, \quad \ell = 3 - k$$

Theorem If there exists a mixed Nash equilibrium,
- if $\sigma^1 \sigma^2 < 0$, the trajectories are all periodical,
- if $\sigma^1 \sigma^2 > 0$, (p^1, p^2) is a saddle. There are two (diagonally opposite) pure Nash equilibria which are asymptotically stable.

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See L. Samuelson : “Evolutionary Games and Equilibrium Selection”

Joint interest

Assume Wolves and Lynxes share a common foe : Man. Then each one benefits from the presence of the other one in repelling the foe. This creates a joint interest (similar to inclusive fitness in E.S.S.)

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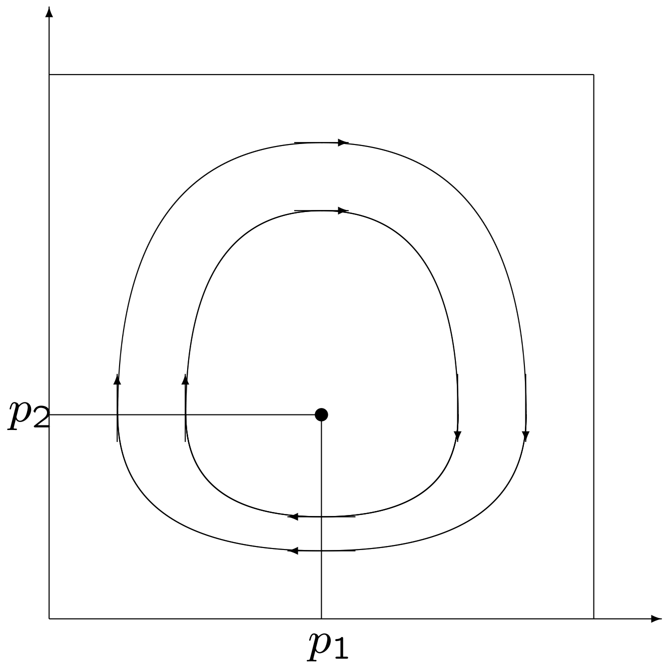
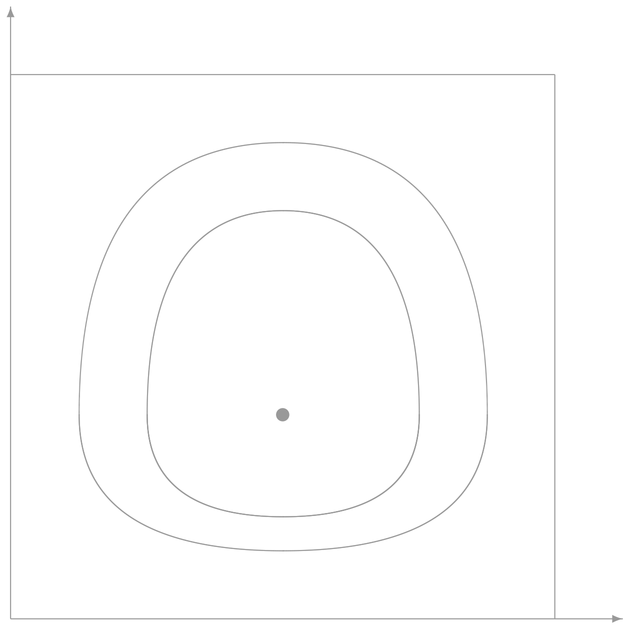
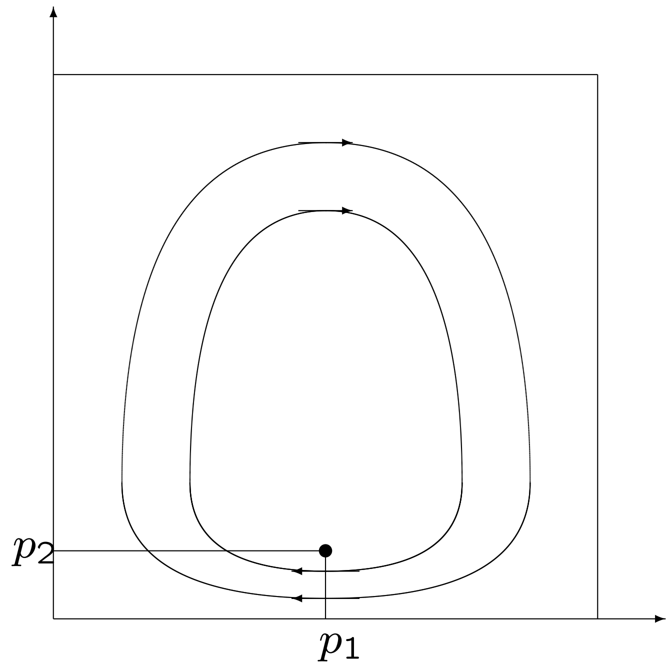
Replaces the game matrices G^k by $G^k_\alpha = (1-\alpha)G^k + \alpha(G^\ell)^t$.

A bifurcation from periodic behaviour to a stable pure strategy occurs as a p^k crosses one or zero.


p_2
p_1

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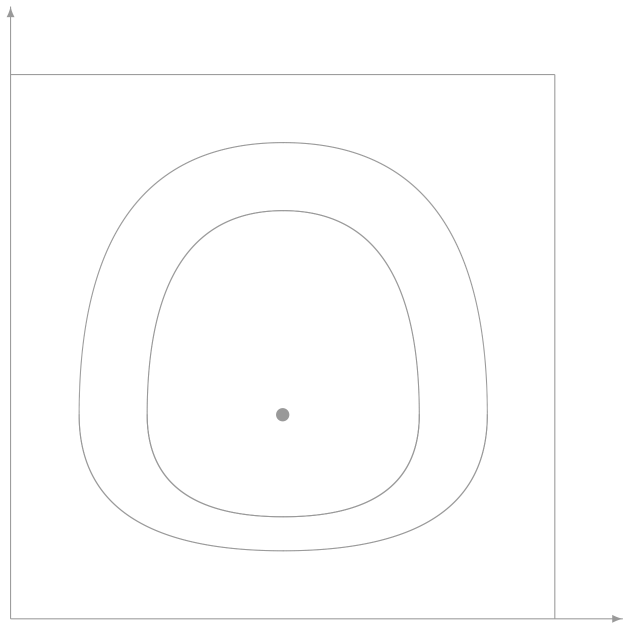

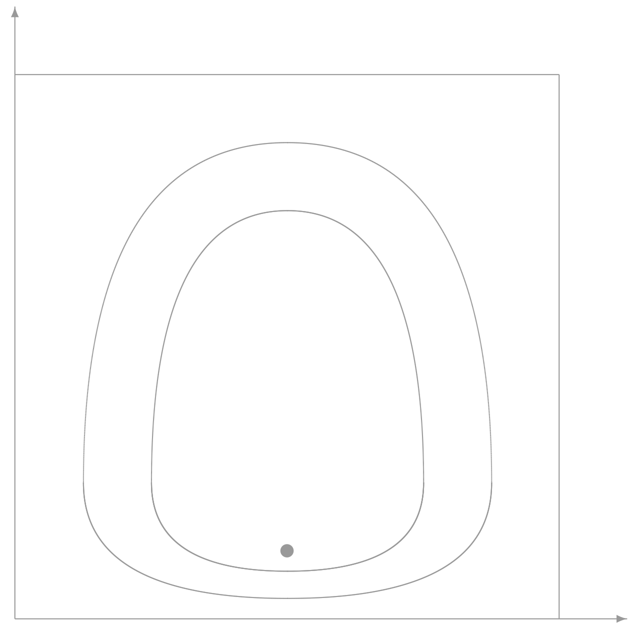

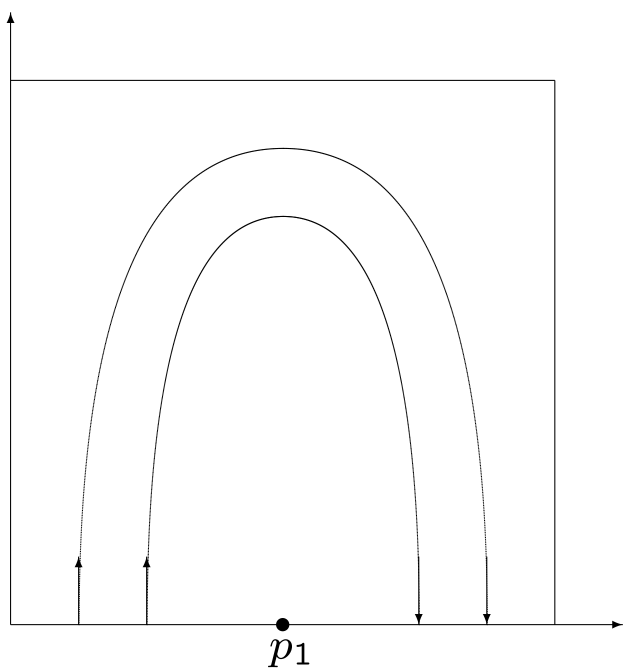
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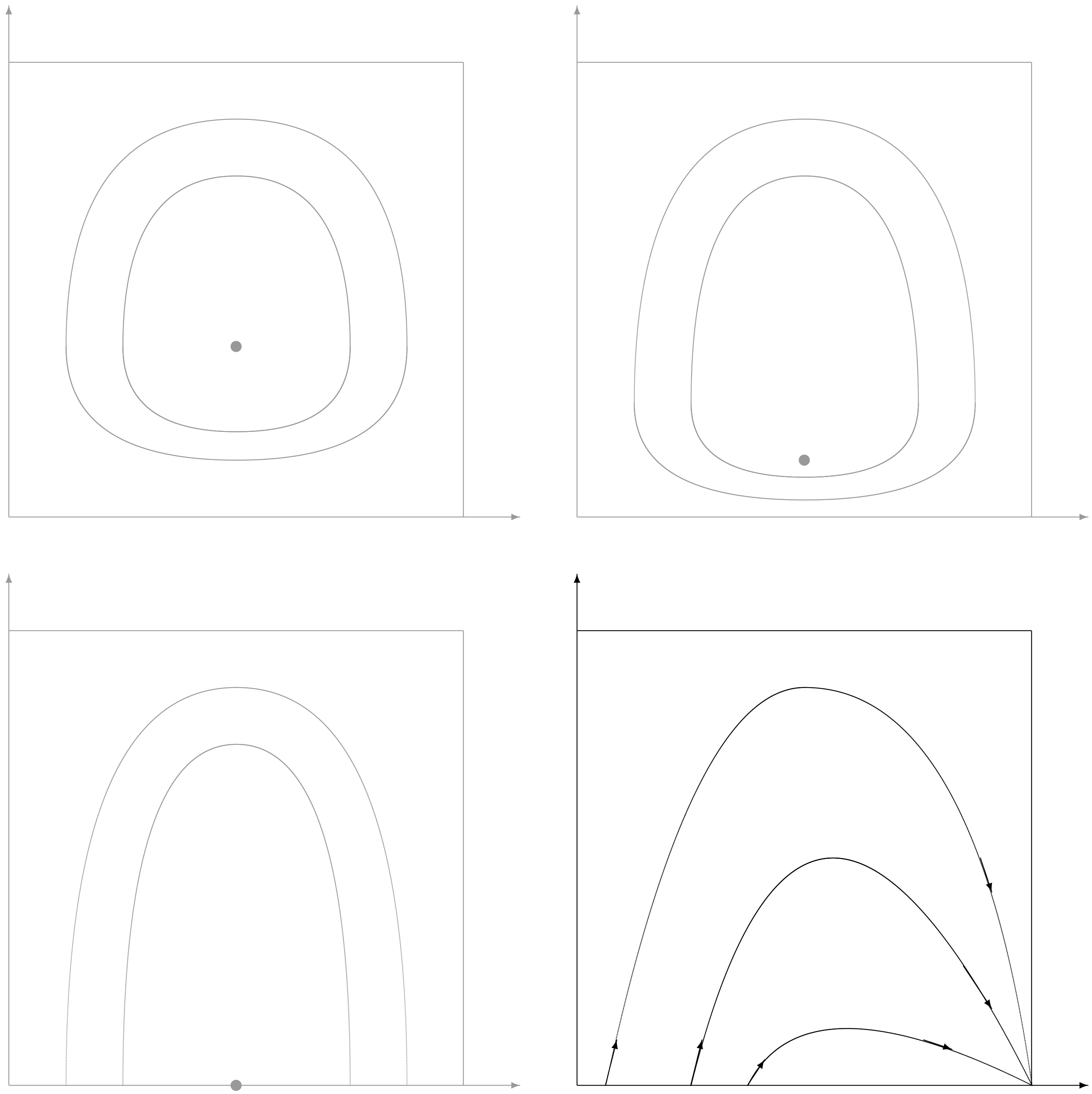

Dynamics in the generating function

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x is a control function, X is of infinite dimension.

Bang-bang control

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State of the world $y \in \mathbb{R}^n$ driven by the population over $[0, T]$:

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Increment of fitness gained by using $x = 1$: $g(y)$. Hence

$$G(x(\cdot), q(\cdot)) = \int_0^T x(t) g(y(t))\, \mathrm{d}t\,.$$

Notation and hypotheses

Assume

$$\mathsf{D}g(y)f(y,0) > 0\,, \quad \mathsf{D}g(y)f(y,1) < 0\,.$$

Let

$$\mathsf{D}_1 f(y,q) = A(y,q)\,, \quad \mathsf{D}_2 f(y,q) = b(y,q)\,, \quad \mathsf{D}g(y) = c(y)\,.$$

Hypothesis More use of the resource depletes it : $c(y)b(y,q) < 0$. $\Rightarrow$

The equation $\dot{g}(y) = c(y)f(y,q) = 0$ generates an implicit function $q = \phi_0(y)$.

Wardrop equilibrium

Wardrop equilibrium $p(\cdot)$ generating a trajectory $z(\cdot)$ is given by $p(t) = \phi(z(t))$ where

$$\phi(y) = \begin{cases} 0 & \text{if } g(y) < 0\,, \\ \phi_0(y) & \text{if } g(y) = 0\,, \\ 1 & \text{if } g(y) > 0\,. \end{cases}$$

The trajectory $z(\cdot)$ reaches $\{y \mid g(y) = 0\}$ at t_0 and stays on it.

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E.S.S. ?

Necessary condition

Let $\Phi(t,s)$ be the tansition matrix of $A(z(\cdot),p(\cdot))$, and

$$h(t,s) := c(z(t))\Phi(t,s)b(z(s),p(s))$$

Theorem A necessary condition for $p(\cdot)$ to be an E.S.S. is that

$$\forall (s,t) \in \mathcal{T} = \{s \le t \in [t_0, T]\}\,, \quad h(t,s)^2 - h(s,s)h(t,t) \le 0\,.$$

Proof Apply the necessary condition $\langle r, H_2 r\rangle < 0$

$$\langle r, H_2 r\rangle = \varepsilon \iint_{\mathcal{T}} r(t)h(t,s)r(s)\mathrm{d}t\,\mathrm{d}s$$

and let $r(\cdot)$ be composed of two strong variations.

The tragedy of the Commons

The shepherds of a village share a common pasture. They may feed their flocks on the pasture ($x = 1$) or refrain ($x = 0$). The grass obeys a logistic law

$$\dot{y} = \alpha \left(1 - \frac{y}{K}\right) y + bq + c\,, \qquad b < 0\,.$$

The “cost” of feeding on the pasture is γ per unit time, and $g(y) = y - \gamma$.

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The equilibrium state is $z = \gamma$, $\quad \phi_0(z) = [(1 - \gamma/K)\gamma + c]/(-b)$.

The tragedy of the Commons

The shepherds of a village share a common pasture. They may feed their flocks on the pasture ($x = 1$) or refrain ($x = 0$). The grass obeys a logistic law

$$\dot{y} = \alpha \left(1 - \frac{y}{K}\right) y + bq + c\,, \qquad b < 0\,.$$

The "cost" of feeding on the pasture is γ per unit time, and $g(y) = y - \gamma$.

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$A(z, p) = \alpha(1 - 2\gamma/K) := a$. The test is, $\forall s < t$,

$$b^2(\mathrm{e}^{2a(t-s)} - 1) \le 0\,.$$

It succeeds if $a \le 0$, i.e. $\gamma \in [K/2, K]$.

General scalar case

If $y \in \mathbb{R}$, the Wardrop "trajectory" is constant :
$y(t) = z$ such that $g(z) = 0$, and $q(t) = p$ such that $c(z)f(z, p) = 0$.

The condition $\langle r, H_2 r \rangle < 0$ implies the local asymptotic stability of

$$\begin{aligned} \dot{y} &= f(y, q)\,, \\ \varepsilon \dot{q} &= q(1-q)g(y)\,, \end{aligned}$$

in the neighborhood of (z, p).

The second equation above is a kind of shortsighted replicator equation.

A routing problem (Joint work with E. Altmann)

An ad-hoc telecommunication network densely covers an open region Ω of the plane $\mathbb{R}^2$. $\Gamma = \partial\Omega = \mathcal{Q} \cup \mathcal{R}$. Messages flow in Ω through $\mathcal{Q}$ at a given rate $\sigma(y) \geq 0$ and a density $\rho(y) \geq 0$ of messages is generated at each $y \in \Omega$. All messages have to leave through $\mathcal{R}$.

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A routing strategy $p(y)$ is a Wardrop equilibrium if a lone message travelling to $\mathcal{R}$ minimizes the travel time by following $x(y) = p(y)/\|p(y)\|$.

Wardrop equilibrium

Let a message originate in $y_0 \in \mathcal{Q} \cup \Omega$, and reach $\mathcal{R}$ in y_1. Let s be the curvilinear abscissa along the path.

$$\frac{\mathrm{d}y}{\mathrm{d}s} = x(s)\,, \qquad G = \int_{y_0}^{y_1} \tau(y(s)) \|q(y(s))\| \, \mathrm{d}s\,.$$

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The H.-J.-B. equation of the optimization problem is

$$\begin{aligned} &\forall y \in \Omega\,, \quad \min_{\|x\|=1} \langle \nabla V(y), x \rangle + \tau(y) \|p(y)\| = 0\,, \\ &\forall y \in \mathcal{R}\,, \quad V(y) = 0\,. \end{aligned}$$

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The optimum is obtained at $x = -\nabla V/\|\nabla V\|$ and $\|\nabla V\| = \tau\|p\|$.

Computing the Wardrop equilibrium

A vector field q is admissible if $\forall y \in \mathcal{Q}$, $\langle q(y), n(y) \rangle = -\sigma(y)$, and $\forall y \in \Omega$, $\operatorname{div} q(y) = \rho(y)$.

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Recapitulating the Wardrop conditions yields

$$
\begin{aligned}
&\forall y \in \mathcal{Q}\,, \quad \langle \nabla V(y), p(y)\rangle = -\sigma(y)\,, \\
&\forall y \in \mathcal{R}\,, \quad V(y) = 0 \\
&\forall y \in \Omega\,, \quad \mathrm{div}\left(\frac{1}{\tau(y)}\nabla V(y)\right) = \rho(y)\,.
\end{aligned}
$$

A classical mixed Dirichlet-Neuman elliptic P.D.E.

Thank you