Explicit formulas for repeated games with absorbing states

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Dynamic Games, Differential Games III,
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Contents

Introduction : value

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- Kohlberg (1974) introduced the operator approach and proved the existence of the asymptotic value $v := \lim_{\lambda \to 0} v(\lambda)$ in the subclass of *absorbing games.*
- The operator approach has been extended by Rosenberg and Sorin (2001) in particular to compact-continuous absorbing games. Mertens, Neyman and Rosenberg proved the existence of the uniform value in the compact-continuous case (but not an explicit formula).

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- Our approach extends to the compact-continuous case and allows also to (1) prove the existence of the asymptotic minmax of multi-player absorbing games, (2) provide an explicit formula for the limit and (3) characterize some periodic equilibrium payoffs of a multi-player game as the discount factor goes to zero.

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- The existence of the uniform minmax was proved by Neyman 2005 for any finite stochastic game.

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- with probability $p(i_t, j_t)$ the interaction continues (the situation is repeated at step $t + 1$).
- If the stream of payoffs is $r(t)$, $t = 1, 2, ...$, the λ-discounted-payoff of the game is $\sum_{t=1}^{\infty} \lambda(1-\lambda)^{t-1} r(t)$. Player I maximizes and player J minimizes.

A quitting game example

	C	A
C	0	1^*
A	1^*	0^*

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\begin{aligned}
v_\lambda &= value \left(\begin{array}{c|c|c} & C & A \\ \hline C & (1-\lambda)v_\lambda & 1 \\ \hline A & 1 & 0 \end{array} \right) \\
&= \max_{x\in[0,1]} \min_{y\in[0,1]} \left[xy(1-\lambda)v_\lambda + x(1-y) + y(1-x)\right] \\
&= \min_{y\in[0,1]} \max_{x\in[0,1]} \left[xy(1-\lambda)v_\lambda + x(1-y) + y(1-x)\right].
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Hence:

$$v_\lambda = x_\lambda = y_\lambda = \frac{1-\sqrt{\lambda}}{1-\lambda}.$$

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- $p^*(i,j) = 1 - p(i,j)$ and $f^*(i,j) = [1 - p(i,j)] \times g(i,j)$.
- Let $\varphi : I \times J \longrightarrow [0,1]$,
- For $\alpha \in M_+(I)$, and $j \in J$, let

$$\varphi(\alpha, j) = \sum_{i\in I} \alpha^i \varphi(i,j)$$

Characterization

Theorem

$v(\lambda)$ satisfies

$$v(\lambda) = \max_{x \in \Delta(I)} \min_{j \in J} \frac{\lambda f(x,j) + (1-\lambda) f^*(x,j)}{\lambda p(x,j) + (1-\lambda) p^*(x,j)}$$

and converges to v as λ goes to zero where,

$$v := \sup_{x \in \Delta(I)} \sup_{\alpha \perp x \in M_+(I)} \min_{j \in J}$$

$$\left(\frac{f^*(x,j)}{p^*(x,j)} 1_{\{p^*(x,j) > 0\}} + \frac{f(x,j) + f^*(\alpha,j)}{p(x,j) + p^*(\alpha,j)} 1_{\{p^*(x,j) = 0\}} \right).$$

where $\alpha \perp x$ means that for every i, $x_i > 0 \Rightarrow \alpha_i = 0$.

Proof

If in the λ-discounted game, player I plays the stationary strategy x, and player J plays a pure stationary strategy $j \in J$, the λ-discounted reward $r(\lambda, x, j)$ satisfies:

$$r(\lambda, x, j) = \lambda f(x, j) + (1-\lambda)\, p(x, j) r(\lambda, x, j) + (1-\lambda) f^*(x, j)$$

hence,

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Since the maximizer has a stationary optimal strategy and the minimizers has a pure stationary best reply (Shapley 1953), the formula for $v(\lambda)$ follows.

Proof

Let $w = \lim_{n\to\infty} v(\lambda_n)$ where $\lambda_n \to 0$.

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There exists $x(\lambda_n) \in \Delta(I)$ such that for every $j \in J$,

$$v(\lambda_n) \leq \frac{\lambda_n f(x(\lambda_n), j) + (1-\lambda_n) f^*(x(\lambda_n), j)}{\lambda_n p(x(\lambda_n), j) + (1-\lambda_n) p^*(x(\lambda_n), j)}.$$

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Case 1: $p^*(x, j) > 0$.
Letting λ_n goes to zero implies

$$w = \lim v(\lambda_n) \leq \frac{f^*(x, j)}{p^*(x, j)}.$$

Proof

Case 2: $p^*(x,j) = \sum_{i \in I} x_i p^*(i,j) = 0$.
Thus, $\sum_{i \in S(x)} p^*(i,j) = 0$ where $S(x) = \{i \in I : x^i > 0\}$.

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$$\sum_{i\in I} \frac{x^i(\lambda_n)(1-\lambda_n)}{\lambda_n} p^*(i,j) = \sum_{i\in I} \alpha^i(\lambda_n) p^*(i,j) = p^*(\alpha(\lambda_n), j)$$

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so,

$$w \leq \lim_{n\to\infty} \frac{f(x,j) + f^*(\alpha(\lambda_n),j)}{p(x,j) + p^*(\alpha(\lambda_n),j)}.$$

Consequently, $w \leq v$.

Proof

Construct a strategy for player I in the λ_n-discounted game that guarantees v as $\lambda_n \to 0$.

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Let $r(\lambda_n)$ be the unique real in $[0, 1]$ that satisfies,

$$r(\lambda_n) = \min_{j \in J} \left[\begin{array}{c} \lambda_n [f(x(\lambda_n), j)] + (1 - \lambda_n)(p(x(\lambda_n), j)) \, r(\lambda_n) \\ + (1 - \lambda_n) f^*(x(\lambda_n), j) \end{array} \right].$$

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It is easy to show that $\lim v(\lambda_n) \geq \lim r(\lambda_n) \geq v$

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- $X^k = \Delta(I^k)$, $X = X^1 \times ... \times X^N$, and $M_+ = M_+(I^1) \times ... \times M_+(I^N)$.

Minmax: characterization

For $x \in X$, $j \in J$, $k \in N$ and $\alpha \in M_+$, $\varphi : I \times J \to [0,1]$, is extended multi-linearly as follows:

$$\varphi(x,j) = \sum_{i=(i^1,\ldots,i^N)\in I} x^1_{i^1} \times \ldots \times x^N_{i^N} \varphi(i,j)$$

$$\varphi(\alpha^k, x^{-k}, j) = \sum_{i=(i^1,\ldots,i^N)\in I} x^1_{i^1} \times \ldots \times x^{k-1}_{i^{k-1}} \times \alpha^k_{i^k} \times x^{k+1}_{i^{k+1}} \ldots \times x^N_{i^N} \varphi(i,j$$

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Theorem

$\underline{v}(\lambda) = \min_{x\in X} \max_{j\in J} \frac{\lambda f(x,j)+(1-\lambda)f^*(x,j)}{\lambda f(x,j)+(1-\lambda)f^*(x,j)}$ *and converges as* $\lambda \to 0$ *to*

$$\underline{v} = \inf_{x\in X} \inf_{\alpha\in M_+:\forall k, \alpha^k \perp x^k} \max_{j\in J} \left(\begin{array}{c} \frac{f^*(x,j)}{p^*(x,j)} \mathbf{1}_{\{p^*(x,j)>0\}} \\ + \frac{f(x,j)+\sum_{k=1}^N f^*(\alpha^k, x^{-k}, j)}{p(x,j)+\sum_{k=1}^N p^*(\alpha^k, x^{-k}, j)} \mathbf{1}_{\{p^*(x,j)=0\}} \end{array} \right).$$

Main modification in proof 1

Let $w = \lim_{n\to\infty} v(\lambda_n)$ where $\lambda_n \to 0$.
Let $x(\lambda_n) \to x$ such that for every $j \in J$,

$$v(\lambda_n) \leq \frac{\lambda_n f(x(\lambda_n), j) + f^*(x(\lambda_n), j)}{\lambda_n p(x(\lambda_n), j) + p^*(x(\lambda_n), j)}.$$

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Let $y(\lambda_n) = x(\lambda_n) - x \to 0$. Then,

$$p^*(x(\lambda_n), j) = p^*(x, j) + \sum_{k=1}^{N} p^*(y^k(\lambda_n), x^{-k}, j) + o(\sum_{k=1}^{N} p^*(y^k(\lambda_n), x^{-k}, j))$$

and

$$f^*(x(\lambda_n), j) = f^*(x, j) + \sum_{k=1}^{N} f^*(y^k(\lambda_n), x^{-k}, j) + o(\sum_{k=1}^{N} f^*(y^k(\lambda_n), x^{-k}, j))$$

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- For $(\alpha, \beta) \in M_+(I) \times M_+(J)$ and $\varphi : I \times J \to [0, 1]$ measurable $\varphi(\alpha, \beta) = \int_{I \times J} \varphi(i, j) d\alpha(i) d\beta(j)$.

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- This is the framework of Rosenberg and Sorin 2001, Israel Journal of Math. They proved the existence of $\lim v(\lambda)$ and provided a variational characterization of it using the derivative of the Shapley Operator around $\lambda \approx 0$.

Characterization

Theorem

$$
\begin{aligned}
v &= \sup_{(x,\alpha):\alpha\perp x}\ \inf_{(y,\beta):\beta\perp y}\left(\begin{array}{c}\frac{f^*(x,y)}{p^*(x,y)}\mathbf{1}_{\{p^*(x,y)>0\}}\\ +\frac{f(x,y)+f^*(\alpha,y)+f^*(x,\beta)}{p(x,y)+p^*(\alpha,y)+p^*(x,\beta)}\mathbf{1}_{\{p^*(x,y)=0\}}\end{array}\right)\\
&= \inf_{(y,\beta):\beta\perp y}\ \sup_{(x,\alpha):\alpha\perp x}\left(\begin{array}{c}\frac{f^*(x,y)}{p^*(x,y)}\mathbf{1}_{\{p^*(x,y)>0\}}\\ +\frac{f(x,y)+f^*(\alpha,y)+f^*(x,\beta)}{p(x,y)+p^*(\alpha,y)+p^*(x,\beta)}\mathbf{1}_{\{p^*(x,y)=0\}}\end{array}\right)
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Proof

Consider a subsequence that converges to $\limsup v_\lambda$. Take an optimal strategy $x(\lambda_n)$ in the λ_n-discounted game that converges to some x.

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Consider any strategy of player 2 of the form $y\left(\lambda_n\right) \propto y + \lambda_n \beta$.

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Consider a subsequence that converges to $\limsup v_\lambda$. Take an optimal strategy $x(\lambda_n)$ in the λ_n-discounted game that converges to some x.

Consider any strategy of player 2 of the form $y(\lambda_n) \propto y + \lambda_n \beta$.

This will imply that $v(\lambda_n)$ is smaller than

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Consider a subsequence that converges to $\limsup v_\lambda$. Take an optimal strategy $x(\lambda_n)$ in the λ_n-discounted game that converges to some x.
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Also, $\liminf v_\lambda \geq \inf \sup$. A trivial comparison principle implies that $\sup \inf \leq \inf \sup$. This end the proof.

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- It would be nice to find an elegant proof for the existence of the uniform value from its formula (Mertens, Neyman and Rosenberg proved existence in the compact-continuous case, to appear in MOR).
- Uniform equilibria of non zero sum absorbing games are much more difficult to study (Paris Mach of Sorin and the existence result of Solan for 3 player games).

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