# Approachability Theory, Discriminating Domain and Differential Games

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# Plan

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# Qualitative Differential Games

- Consider the two controlled dynamical system

$$\begin{cases} \dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), \mathbf{v}(t)) & \text{for a.e } t \geq 0 \\ \mathbf{x}(0) = x_0 \in \mathbb{R}^d, \end{cases}$$

- Controls of Player 1 : measurable maps $u : [0, +\infty) \mapsto U$.
- Controls of Player 2: measurable maps $v : [0, +\infty) \mapsto V$.
- Let $K \subset \mathbb{R}^d$ be a nonempty closed set.
- The goal of Player 1 is to keep the state of the system in $K$ indefinitely while Player 2 wants to make the state reach $K^c$.

# Plan

## Nonanticpative strategies

A map $\alpha : \mathcal{V} \rightarrow \mathcal{U}$ is a **nonanticipative strategy (NA)** if $\forall\, \mathbf{v}_1 \in \mathcal{V}$, $\forall \mathbf{v}_2 \in \mathcal{V}$ we have $\mathbf{v}_1 = \mathbf{v}_2$ on $[0, t]$ Then

$$\alpha(\mathbf{v}_1)(s) \equiv \alpha(\mathbf{v}_2)(s) \quad \forall s \in [0, t].$$

## Discriminating domain (Aubin '89)

$K$ is a discriminating domain if for every $x_0 \in K$, $\exists\, \alpha \in NA(\mathcal{V}, \mathcal{U})$, s.t $\forall \mathbf{v} \in \mathcal{V}$, $\forall t > 0$ $\mathbf{x}[x_0, \alpha(\mathbf{v}), \mathbf{v}](t) \in K$.
The strategy $\alpha$ is then said to preserve $K$.

## Proximal normals

$NP_K(x) = \{p \in \mathbb{R}^d / d_K(x+p) = \|p\|\}$ is the of proximal normal set.

## Interpretation Theorem (Cardaliaguet '96)

$K$ is a discriminating domain if and only if: $\forall x \in K, \quad \forall p \in NP_K(x)$

$$\sup_{v \in V} \inf_{u \in U} \langle f(x,u,v), p \rangle \leq 0.$$

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# Plan

1. Repeated games with vectorial payoff
   - Set up
   - Characterization of Approachability

# Repeated games with vectorial payoff

- Let $A = a_{i,j}$ be an $I \times J$ matrix with coefficients in $\mathbb{R}^d$
- 2 players matrix game repeated indefinitely.
- At each stage $n = 1, 2, \ldots$, each player chooses an element in his set of actions: $i_n \in I$ for Player 1 (resp. $j_n \in J$ for Player 2), the corresponding outcome is $g_n = A_{i_n j_n} \in \mathbb{R}^d$ and the couple of actions $(i_n, j_n)$ is announced to both players.
- The average payoff $\overline{g}_n = \frac{1}{n} \sum_{m=1}^{n} g_m$
- The goal of Player 1 is to reach a closed target $K$ while the goal of Player 2 is to avoid it.

- $H_n = (I \times J)^n$ is the set of possible histories up to stage $n$.
- $\Sigma$ is the set of **strategies** of Player 1 namely mappings:

$$H = \cup_n H_n \mapsto \Delta(I)$$

  meaning that: If at stage $n$, the history is $h_{n-1} \in H_{n-1}$, Player 1 chooses an action in $I$ according to the probability distribution $\sigma(h_{n-1}) \in \Delta(I)$
- $\mathcal{T}$ is the set of strategies of Player 2 defined similarly.

### Approachability (Blackwell '56)

A nonempty closed set $K$ in $\mathbb{R}^d$ is **approachable** for Player 1 if, for every $\varepsilon > 0$, there exists a strategy $\sigma$ of Player 1 and $N \in \mathbb{N}$ such that, for any strategy $\tau$ of Player 2 and any $n \geq N$

$$\mathbb{E}_{\sigma,\tau}(d_K(\overline{g}_n)) \leq \varepsilon.$$

# Plan

# Characterization of Approachability

**B-set (Blackwell '56)**

A closed set $K$ in $\mathbb{R}^d$ is a **B**-set for Player 1 if for any $z \notin K$, there exists a closest point $x$ to $K$ and a mixed action $u = u(z)$ in $U = \Delta(I)$ such that the hyperplane through $x$ orthogonal to the segment $[xz]$ separates $z$ from $P(u) = \{uAv | v \in V = \Delta(J)\}$. Namely:

$$\exists u \in U \, \forall v \in V \qquad \langle z - x, uAv - x \rangle \leq 0.$$

**Theorem (Blackwell '56)**

Every **B**-Set for Player 1 is approachable by that player.

**Theorem (Hou '71, Spinat '00)**

If a nonempty closed set is approachable then it contains a **B**-set.

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2. From repeated to differential games
   - The "expected" repeated game $G^\star$

# The "expected" repeated game $G^\star$

- Consider $A$ the same matrix as defined before.
- 2 players.
- At each stage $n$, each player chooses an element $u_n$ in his set of actions: $u_n \in U$ for Player 1 (resp. $v_n \in V$ for Player 2), the corresponding outcome is $g_n^\star = u_n A v_n \in \mathbb{R}^d$
- The couple of actions $(u_n, v_n)$ is announced.
- The payoff up to stage $n$ is the average payoff over the last stages $\overline{g}_n^\star = \frac{1}{n} \sum_{m=1}^{n} g_m^\star$.

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- $H_n^\star = (U \times V)^n$ is the set of possible histories up to stage $n$.
- $\Sigma$ (resp $\mathcal{T}$) is the set of **strategies** of Player 1 namely mappings :

$$H^\star = \cup_n H_n^\star \mapsto U$$

(resp. Player 2 $H^\star \mapsto V$).

## $\star$-Approachability

A nonempty closed set $K$ in $\mathbb{R}^d$ is $\star$approachable for Player 1 if, for every $\varepsilon > 0$, there exists a strategy $\sigma^\star$ of Player 1 and $N \in \mathbb{N}$ such that, for any strategy $\tau^\star$ of Player 2 and any $n \geq N$

$$d_K(\overline{g}_n^\star) \leq \varepsilon.$$

- Notice that if $K$ is a **B**-set then it is $\star$approachable.

# The related differential game $\Gamma$

- We mimic the average payoff $\overline{g}_n^\star$ by a continuous time average payoff, denoted by $\widetilde{g}$, with $\widetilde{g}(0) = 0$ and for $t > 0$ $\widetilde{g}[\mathbf{u}, \mathbf{v}](t) = \frac{1}{t} \int_0^t \mathbf{u}(s) A \mathbf{v}(s) ds$.
- $$\frac{\partial \widetilde{g}[\mathbf{u}, \mathbf{v}](t)}{\partial t} = \frac{-\widetilde{g}(t) + \mathbf{u}(t) A \mathbf{v}(t)}{t}.$$
- Set $t = e^s$, and denote $\mathbf{x}(s) = \widetilde{g}(e^s)$
$$\dot{\mathbf{x}}(s) = -\mathbf{x}(s) + \mathbf{u}(s) A \mathbf{v}(s).$$
- which is the dynamics of a qualitative differential with $f(x, u, v) = -x + uAv$.

# Main results

**Theorem**

A nonempty closed set $K \subset \mathbb{R}^d$ is a discriminating domain in the differential game $\Gamma$ for Player 1 if and only if $K$ is a **B**-set for Player 1 in $G$ (or $G^\star$).

- Suppose that $K$ is a **B**-Set for Player 1. Let $x \in K$ and $p \in NP_K(x)$ s.t $p \neq 0$.
- let $z = x + p/2$ and observe that $\pi_K(z)$ is reduced to the singleton $\{x\}$.
- Hence Since $K$ is **B**-set there exists a mixed move $u \in U$ such that, for every $v \in V$,

$$\langle uAv - x, z - x \rangle \leq 0.$$

- $$\sup_{v \in V} \inf_{u \in U} \langle f(x,u,v), p \rangle \leq \inf_{u \in U} \sup_{v \in V} \langle f(x,u,v), p \rangle \leq 0.$$

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## Proposition

If a nonempty closed set $K \subset \mathbb{R}^d$ is a **B**-set for Player 1 in $G$ or $G^\star$, there exists a NA strategy $\alpha$ of Player 1 in $\Gamma$, such that for every $\mathbf{v} \in \mathcal{V}$

$$\forall t \geq 1 \quad d_K(\widetilde{g}[\alpha(\mathbf{v}), \mathbf{v}](t)) \leq \frac{M}{t}.$$

- Given $y_0 \in K$ and $x_0 \notin K$, let Player 1 use the related preserving strategy $\alpha$.
- Denote $y_s = \mathbf{x}[y_0, \alpha(\mathbf{v}), \mathbf{v}](s)$ and $x_s = \mathbf{x}[x_0, \alpha(\mathbf{v}), \mathbf{v}](s)$.
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# Main Theorem

**Theorem**

A closed set $K$ is $\star$-approachable for Player 1 if and only if it contains a **B**-Set for that player.

**Corollary**

Approachability and $\star$-approachable do coincide.

# From preserving NA strategies to approachability strategies

**Theorem**

For any $\varepsilon > 0$ and any non-anticipative strategy $\alpha$ preserving $K$ in the differential game $\Gamma$, there exists an approachability strategy $\sigma$ for $K + \varepsilon B(0,1)$ in the repeated game $G$.

## Nonanticpative strategies with delay

We say that a map $\delta : \mathcal{U} \mapsto \mathcal{V}$ is a **nonanticipative strategy with delay (NAD)** if there exits a subdivision of time $t_1 < t_2 < .. < t_n < ..$ for which we have the following property :

$$\forall \;\; w_1, w_2 \in \mathcal{U} \;\; s.t \;\; w_1(s) \equiv w_2(s) \;\; for \; a.e \; s \in [0, t_i]$$

$$Then, \;\; \delta(w_1)(s) \equiv \delta(w_2)(s) \;\; for \; a.e \; s \in [0, t_{i+1}].$$

The idea of the construction is the following:

- Given a NA strategy $\alpha$ we will show that it can be approximated in term of range by a piecewise constant NAD strategy $\bar{\alpha}$.
- When applied to $\alpha$ preserving $K$ (hence approaching $K$), we will obtain a NAD strategy approaching $K$.
- Starting from the repeated game $G^*$ this procedure will produce an approachability strategy.

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