Approachability Theory, Discriminating Domain and Differential Games

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Plan

0 Qualitative Differential Games
1 Repeated games with vectorial payoff
2 From repeated to differential games
3 B-set and Discriminating domain
4 Approachability strategies vs preserving strategies
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Plan

Qualitative Differential Games
- Set up
- Discriminating Domains
Consider the two controlled dynamical system

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t), v(t)) \quad \text{for a.e } t \geq 0 \\
x(0) &= x_0 \in \mathbb{R}^d,
\end{align*}
\]

Controls of Player 1: measurable maps \(u: [0, +\infty) \mapsto U\).

Controls of Player 2: measurable maps \(v: [0, +\infty) \mapsto V\).

Let \(K \subset \mathbb{R}^d\) be a nonempty closed set.

The goal of Player 1 is to keep the state of the system in \(K\) indefinitely while Player 2 wants to make the state reach \(K^c\).
Plan

Qualitative Differential Games
- Set up
- Discriminating Domains
Nonanticipative strategies

A map $\alpha : \mathcal{V} \rightarrow \mathcal{U}$ is a **nonanticipative strategy (NA)** if

$\forall v_1 \in \mathcal{V}, \forall v_2 \in \mathcal{V}$ we have $v_1 = v_2$ on $[0, t]$ Then

$$\alpha(v_1)(s) \equiv \alpha(v_2)(s) \ \forall s \in [0, t].$$

Discriminating domain (Aubin '89)

$K$ is a discriminating domain if for every $x_0 \in K$, $\exists \alpha \in NA(\mathcal{V}, \mathcal{U})$, s.t

$\forall v \in \mathcal{V}, \forall t > 0 \ x[x_0, \alpha(v), v](t) \in K.$

The strategy $\alpha$ is then said to preserve $K$. 
Proximal normals

\[ NP_K(x) = \{ p \in \mathbb{R}^d / d_K(x + p) = \|p\| \} \] is the of proximal normal set.

Interpretation Theorem (Cardaliaguet ’96)

\( K \) is a discriminating domain if and only if: \( \forall x \in K, \ \forall p \in NP_K(x) \)

\[
\sup_{v \in V} \inf_{u \in U} \langle f(x, u, v), p \rangle \leq 0.
\]
Proximal normals

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Interpretation Theorem (Cardaliaguet '96)

\[ K \text{ is a discriminating domain if and only if: } \forall x \in K, \ \forall p \in NP_K(x) \]
\[ \sup_{\nu \in \mathcal{V}} \inf_{u \in \mathcal{U}} \langle f(x, u, \nu), p \rangle \leq 0. \]
Plan

1. Repeated games with vectorial payoff
   - Set up
   - Characterization of Approachability
Repeated games with vectorial payoff

- Let $A = a_{i,j}$ be an $I \times J$ matrix with coefficients in $\mathbb{R}^d$
- 2 players matrix game repeated indefinitely.
- At each stage $n = 1, 2, ..., $ each player chooses an element in his set of actions: $i_n \in I$ for Player 1 (resp. $j_n \in J$ for Player 2), the corresponding outcome is $g_n = A_{i_nj_n} \in \mathbb{R}^d$ and the couple of actions $(i_n, j_n)$ is announced to both players.
- The average payoff $\bar{g}_n = \frac{1}{n} \sum_{m=1}^{n} g_m$
- The goal of Player 1 is to reach a closed target $K$ while the goal of Player 2 is to avoid it.
• \( H_n = (I \times J)^n \) is the set of possible histories up to stage \( n \).
• \( \Sigma \) is the set of strategies of Player 1 namely mappings:

\[
H = \bigcup_n H_n \mapsto \Delta(I)
\]

meaning that: If at stage \( n \), the history is \( h_{n-1} \in H_{n-1} \), Player 1 chooses an action in \( I \) according to the probability distribution \( \sigma(h_{n-1}) \in \Delta(I) \)

• \( T \) is the set of strategies of Player 2 defined similarly.

**Approachability (Blackwell ’56)**

A nonempty closed set \( K \) in \( \mathbb{R}^d \) is **approachable** for Player 1 if, for every \( \varepsilon > 0 \), there exists a strategy \( \sigma \) of Player 1 and \( N \in \mathbb{N} \) such that, for any strategy \( \tau \) of Player 2 and any \( n \geq N \)

\[
\mathbb{E}_{\sigma,\tau}(d_K(\bar{g}_n)) \leq \varepsilon.
\]
1. Repeated games with vectorial payoff
   - Set up
   - Characterization of Approachability
Characterization of Approachability

**B-set (Blackwell ’56)**

A closed set $K$ in $\mathbb{R}^d$ is a B-set for Player 1 if for any $z \notin K$, there exists a closest point $x$ to $K$ and a mixed action $u = u(z)$ in $U = \Delta(I)$ such that the hyperplane through $x$ orthogonal to the segment $[xz]$ separates $z$ from $P(u) = \{uAv | v \in V = \Delta(J)\}$. Namely:

$$\exists u \in U \forall v \in V \langle z - x, uAv - x \rangle \leq 0.$$  

**Theorem (Blackwell ’56)**

Every B-Set for Player 1 is approachable by that player.

**Theorem (Hou ’71, Spinat ’00)**

If a nonempty closed set is approachable then it contains a B-set.
Characterization of Approachability

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From repeated to differential games

- The “expected” repeated game $G^*$
The “expected” repeated game \( G^* \)

- Consider \( A \) the same matrix as defined before.
- 2 players.
- At each stage \( n \), each player chooses an element \( u_n \) in his set of actions: \( u_n \in U \) for Player 1 (resp. \( v_n \in V \) for Player 2), the corresponding outcome is \( g_n^* = u_n A v_n \in \mathbb{R}^d \)
- The couple of actions \((u_n, v_n)\) is announced.
- The payoff up to stage \( n \) is the average payoff over the last stages \( \overline{g}_n^* = \frac{1}{n} \sum_{m=1}^n g_m^* \).
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2 players.

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- The couple of actions $(u_n, v_n)$ is announced.
- The payoff up to stage $n$ is the average payoff over the last stages $\overline{g}_n^* = \frac{1}{n} \sum_{m=1}^{n} g_m^*$. 
- $H_n^* = (U \times V)^n$ is the set of possible histories up to stage $n$.
- $\Sigma$ (resp $T$) is the set of strategies of Player 1 namely mappings:

$$H^* = \bigcup_n H_n^* \mapsto U$$

(resp. Player 2 $H^* \mapsto V$).

**\$\star\$-Approachability**

A nonempty closed set $K$ in $\mathbb{R}^d$ is $\star$-approachable for Player 1 if, for every $\varepsilon > 0$, there exists a strategy $\sigma^*$ of Player 1 and $N \in \mathbb{N}$ such that, for any strategy $\tau^*$ of Player 2 and any $n \geq N$

$$d_K(\bar{g}^*_n) \leq \varepsilon.$$  

- Notice that if $K$ is a $\mathcal{B}$-set then it is $\star$-approachable.
The related differential game $\Gamma$

- We mimic the average payoff $\bar{g}^*_n$ by a continuous time average payoff, denoted by $\tilde{g}$, with $\tilde{g}(0) = 0$ and for $t > 0$
  \[ \tilde{g}[u, v](t) = \frac{1}{t} \int_0^t u(s)A_v(s)ds. \]

- \[
  \frac{\partial \tilde{g}[u, v](t)}{\partial t} = -\tilde{g}(t) + \frac{u(t)A_v(t)}{t}. 
  \]

- Set $t = e^s$, and denote $x(s) = \tilde{g}(e^s)$
  \[
  \dot{x}(s) = -x(s) + u(s)A_v(s). 
  \]

- which is the dynamics of a qualitative differential with
  \[ f(x, u, v) = -x + uA_v. \]
Main results

Theorem

A nonempty closed set \( K \subset \mathbb{R}^d \) is a discriminating domain in the differential game \( \Gamma \) for Player 1 if and only if \( K \) is a \( \mathcal{B} \)-set for Player 1 in \( G \) (or \( G^* \)).

- Suppose that \( K \) is a \( \mathcal{B} \)-Set for Player 1. Let \( x \in K \) and \( p \in NP_K(x) \) s.t \( p \neq 0 \).
- let \( z = x + p/2 \) and observe that \( \pi_K(z) \) is reduced to the singleton \( \{x\} \).
- Hence Since \( K \) is \( \mathcal{B} \)-set there exists a mixed move \( u \in U \) such that, for every \( v \in V \),
  \[
  \langle uAv - x, z - x \rangle \leq 0.
  \]
- 
  \[
  \sup_{v \in V} \inf_{u \in U} \langle f(x, u, v), p \rangle \leq \inf_{u \in U} \sup_{v \in V} \langle f(x, u, v), p \rangle \leq 0.
  \]

Thus, \( K \) is a discriminating domain for Player 1.
Main results

**Theorem**

A nonempty closed set $K \subset \mathbb{R}^d$ is a discriminating domain in the differential game $\Gamma$ for Player 1 if and only if $K$ is a $\mathcal{B}$-set for Player 1 in $G$ (or $G^*$).

- Suppose that $K$ is a $\mathcal{B}$-Set for Player 1. Let $x \in K$ and $p \in NP_K(x)$ s.t $p \neq 0$.
- let $z = x + p/2$ and observe that $\pi_K(z)$ is reduced to the singleton $\{x\}$.
- Hence Since $K$ is $\mathcal{B}$-set there exists a mixed move $u \in U$ such that, for every $v \in V$,

$$\langle uAv - x, z - x \rangle \leq 0.$$ 

$$\sup_{v \in V} \inf_{u \in U} \langle f(x, u, v), p \rangle \leq \inf_{u \in U} \sup_{v \in V} \langle f(x, u, v), p \rangle \leq 0.$$ 

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  $$\sup_{v \in V} \inf_{u \in U} \langle f(x, u, v), p \rangle \leq \inf_{u \in U} \sup_{v \in V} \langle f(x, u, v), p \rangle \leq 0.$$ 

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Thus, $K$ is a discriminating domain for Player 1.
Main results

Theorem

A nonempty closed set $K \subset \mathbb{R}^d$ is a discriminating domain in the differential game $\Gamma$ for Player 1 if and only if $K$ is a $B$-set for Player 1 in $G$ (or $G^*$).

- Suppose that $K$ is a $B$-Set for Player 1. Let $x \in K$ and $p \in NP_K(x)$ s.t $p \neq 0$.
- Let $z = x + p/2$ and observe that $\pi_K(z)$ is reduced to the singleton $\{x\}$.
- Hence Since $K$ is $B$-set there exists a mixed move $u \in U$ such that, for every $v \in V$,
  \[ \langle uA v - x, z - x \rangle \leq 0. \]

Thus, $K$ is a discriminating domain for Player 1.

\[ \sup_{v \in V} \inf_{u \in U} \langle f(x, u, v), p \rangle \leq \inf_{u \in U} \sup_{v \in V} \langle f(x, u, v), p \rangle \leq 0. \]
Proposition

If a nonempty closed set $K \subset \mathbb{R}^d$ is a $B$-set for Player 1 in $G$ or $G^*$, there exists a NA strategy $\alpha$ of Player 1 in $\Gamma$, such that for every $\mathbf{v} \in \mathcal{V}$

$$\forall t \geq 1 \quad d_K(\tilde{g}[\alpha(\mathbf{v}), \mathbf{v}](t)) \leq \frac{M}{t}.$$ 

- Given $y_0 \in K$ and $x_0 \notin K$, let Player 1 use the related preserving strategy $\alpha$.
- Denote $y_s = x[y_0, \alpha(\mathbf{v}), \mathbf{v}](s)$ and $x_s = x[x_0, \alpha(\mathbf{v}), \mathbf{v}](s)$. Since $\dot{x}_t = \alpha(\mathbf{v})(t)A\mathbf{v}(t) - x_t$, $\dot{y}_t = \alpha(\mathbf{v})(t)A\mathbf{v}(t) - y_t$, one has
  $$\frac{d}{dt}(x_t - y_t) = \dot{x}_t - \dot{y}_t = -(x_t - y_t)$$

- Hence $\|x_t - y_t\| = \|x_0 - y_0\|e^{-t}$
- Since $y_t \in K$
  $$d_K(x_t) \leq \|x_0 - y_0\|e^{-t}.$$
Proposition

If a nonempty closed set $K \subset \mathbb{R}^d$ is a $\mathbf{B}$-set for Player 1 in $G$ or $G^*$, there exists a NA strategy $\alpha$ of Player 1 in $\Gamma$, such that for every $\mathbf{v} \in \mathcal{V}$

\[ \forall t \geq 1 \quad d_K(\tilde{g}[\alpha(\mathbf{v}), \mathbf{v}](t)) \leq \frac{M}{t}. \]

- Given $y_0 \in K$ and $x_0 \not\in K$, let Player 1 use the related preserving strategy $\alpha$.
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If a nonempty closed set \( K \subset \mathbb{R}^d \) is a \( \mathcal{B} \)-set for Player 1 in \( G \) or \( G^* \), there exists a NA strategy \( \alpha \) of Player 1 in \( \Gamma \), such that for every \( \mathbf{v} \in \mathcal{V} \)

\[
\forall t \geq 1 \quad d_K(\tilde{g}[\alpha(\mathbf{v}), \mathbf{v}](t)) \leq \frac{M}{t}.
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- Given \( y_0 \in K \) and \( x_0 \notin K \), let Player 1 use the related preserving strategy \( \alpha \).
- Denote \( y_s = x[y_0, \alpha(\mathbf{v}), \mathbf{v}](s) \) and \( x_s = x[x_0, \alpha(\mathbf{v}), \mathbf{v}](s) \).
  Since \( \dot{x}_t = \alpha(\mathbf{v})(t)A_{\mathbf{v}}(t) - x_t, \quad \dot{y}_t = \alpha(\mathbf{v})(t)A_{\mathbf{v}}(t) - y_t \)
  one has

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\frac{d}{dt}(x_t - y_t) = \dot{x}_t - \dot{y}_t = -(x_t - y_t)
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- Denote $y_s = x[y_0, \alpha(v), v](s)$ and $x_s = x[x_0, \alpha(v), v](s)$.
  
  Since $\dot{x}_t = \alpha(v)(t)A v(t) - x_t$,  \quad $\dot{y}_t = \alpha(v)(t)A v(t) - y_t$,

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Main Theorem

Theorem
A closed set $K$ is $\star$-approachable for Player 1 if and only if it contains a $B$-Set for that player.

Corollary
Approachability and $\star$-approachable do coincide.
Theorem

For any $\varepsilon > 0$ and any non-anticipative strategy $\alpha$ preserving $K$ in the differential game $\Gamma$, there exists an approachability strategy $\sigma$ for $K + \varepsilon B(0, 1)$ in the repeated game $G$. 
We say that a map $\delta : \mathcal{U} \mapsto \mathcal{V}$ is a \textbf{nonanticipative strategy with delay (NAD)} if there exits a subdivision of time $t_1 < t_2 < \ldots < t_n < \ldots$ for which we have the following property:

$$\forall \ w_1, w_2 \in \mathcal{U} \text{ s.t. } w_1(s) \equiv w_2(s) \text{ for a.e } s \in [0, t_i]$$

Then, $\delta(w_1)(s) \equiv \delta(w_2)(s) \text{ for a.e } s \in [0, t_{i+1}].$

The idea of the construction is the following:

- Given a NA strategy $\alpha$ we will show that it can be approximated in term of range by a piecewise constant NAD strategy $\bar{\alpha}$.
- When applied to $\alpha$ preserving $K$ (hence approaching $K$), we will obtain a NAD strategy approaching $K$.
- Starting from the repeated game $G^*$ this procedure will produce an approachability strategy.
Nonanticipative strategies with delay

We say that a map $\delta : \mathcal{U} \mapsto \mathcal{V}$ is a nonanticipative strategy with delay (NAD) if there exists a subdivision of time $t_1 < t_2 < .. < t_n < ..$ for which we have the following property:

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